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I.—ON THE LUNAR THEORY.

Continued from vol. III. p. 257.

15. WE shall now apply the equations of Article (12) to the Lunar Theory, in the following manner. We shall first suppose that the moon moves in the plane of the ecliptic, and afterwards prove the results obtained on this hypothesis to be true when the inclination is taken into account; we shall then determine the motion of the plane of the orbit, the mean motions of the perigee and node to the third order, and the parallactic inequality.

By Article (12), we have

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{\mu}{r^2} + \frac{dR}{dr} \quad \ldots\ldots\ldots\ldots (1)$$

and

$$\frac{dh}{dt} = \frac{dR}{d\theta} \quad \ldots\ldots\ldots\ldots\ldots (2);$$

and by Article (3), we have

$$\frac{d^2r}{dt^2} = -\frac{d^2u}{d\theta^2} h^2u^2 - \frac{du}{d\theta}\frac{dh}{dt}.$$

Therefore (1) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} - \frac{r^2}{h^2}\frac{dR}{dr} + \frac{1}{h^2}\frac{dr}{d\theta}\frac{dR}{d\theta} \quad \ldots\ldots (3);$$

and since $\frac{d\theta}{dt} = \frac{h}{r^2}$ (2) becomes

$$\frac{d(h^2)}{d\theta} = 2r^2\frac{dR}{d\theta} \quad \ldots\ldots\ldots\ldots (4).$$

16. To determine R, we have, by Article (14), putting $\cos(r, r') = p$,

$$R = m'\left(\frac{1}{r_1} - \frac{r}{r'^2}p\right),$$

and $\frac{1}{r_1} = \frac{1}{r'}(1 - 2pz + z^2)^{-\frac{1}{2}}$ where $z = \frac{r}{r'}$

$$= \frac{1}{r'}\left\{1 + \tfrac{1}{2}(2pz - z^2) + \frac{1.3}{2.4}(2pz - z^2)^2 + \frac{1.3.5}{2.4.6}(2pz - z^2)^3 \&c. \ldots\right\}$$

$$= \frac{1}{r'}\left\{1 + pz + \frac{3p^2 - 1}{2}z^2 + \frac{5p^3 - 3p}{2}z^3 \&c. \ldots\right\}.$$

Hence, retaining only the lowest power of z or $\frac{r}{r'}$, we have (observing that p or $\cos(r, r') = \cos(\theta - \theta')$, since we suppose the moon to move in the plane of the ecliptic)

$$R = \frac{m'}{r'} + \frac{m'r^2}{r'^3}\cdot\frac{1 + 3\cos 2(\theta - \theta')}{4},$$

and therefore

$$\frac{dR}{dr} = \frac{m'r}{r'^3}\cdot\frac{1 + 3\cos 2(\theta - \theta')}{2} \ldots\ldots (5).$$

$$\frac{dR}{d\theta} = -\frac{m'r^2}{r'^3}\cdot\frac{3\sin 2(\theta - \theta')}{2} \ldots\ldots\ldots (6).$$

We shall expand the second members of (3) and (4) as far as the third order of small quantities, retaining, however, only those terms of the third order in which the coefficient of θ is nearly equal to unity or zero; and this we shall do by substituting for r, r', and θ' their values in terms of θ to the first order, namely

$$r = a\{1 - e\cos(\theta - \omega)\} \quad r' = a'\{1 - e'\cos(\theta' - \omega')\} \ldots (7),$$

$$\theta' = m\theta + \beta + 2e'\sin(m\theta + \beta - \omega') \ldots\ldots (8).$$

17. To determine $\frac{\mu}{h^2}$ in this manner, we have, by (4) and (6),

$$\frac{d(h^2)}{d\theta} = -\frac{3m'r^4}{r'^3}\sin 2(\theta - \theta');$$

or, by (7), putting $\frac{m'a^3}{\mu a'^3} = m^2$,

$$\frac{d(h^2)}{d\theta} = -3a\mu m^2\{1 - 4e\cos(\theta - \omega) + 3e'\cos(\theta' - \omega')\}\sin 2(\theta - \theta').$$

In the terms of this, which are multiplied by e and e', we may put for θ' its first approximate value, namely, $m\theta + \beta$. In the remaining term, if we put for θ' its value (8), expand to the first power of e', and then transform products of sines and cosines into sums, in the usual way, it is evident that $\sin 2(\theta - \theta')$ will contain no term in which the coefficient of θ is nearly equal to unity or zero. It follows therefore that

we may suppose θ' to be equal to $m\theta + \beta$, simply, in all the terms of this equation.

This being the case, the product $\cos(\theta' - \omega')\sin 2(\theta - \theta')$, transformed into a sum, gives no term to be retained, and the product $\cos(\theta - \omega)\sin 2(\theta - \theta')$ gives only $\frac{1}{2}\sin(\theta - 2\theta' + \omega)$: we have therefore

$$\frac{d(h^2)}{d\theta} = -3a\mu m^2\{\sin 2(\theta - \theta') - 2e\sin(\theta - 2\theta' + \omega)\}.$$

Therefore, integrating, supplying the proper constant $\{a\mu(1 - e^2)\}$, and neglecting powers and products of m and e above the second order, we have

$$h^2 = a\mu\left\{1 - e^2 + \frac{3m^2}{2}\cos 2(\theta - \theta') - 6m^2e\cos(\theta - 2\theta' + \omega)\right\},$$

$$\therefore \frac{\mu}{h^2} = \frac{1}{a}\left\{1 + e^2 - \frac{3m^2}{2}\cos 2(\theta - \theta') + 6m^2e\cos(\theta - 2\theta' + \omega)\right\}.$$

18. To find $-\frac{r^2}{h^2}\frac{dR}{dr}$, we have, by (5) and (7),

$$-\frac{r^2}{h^2}\frac{dR}{dr} = -\frac{m'r^3}{a\mu r'^3}\frac{1 + 3\cos 2(\theta - \theta')}{2}$$

$$= -\frac{m^2}{a}\{1 - 3e\cos(\theta - \omega) + 3e'\cos(\theta' - \omega')\}.\frac{1 + 3\cos 2(\theta - \theta')}{2}.$$

For exactly the same reasons as before, we may here suppose θ' to be equal to $m\theta + \beta$ simply; and then, retaining only the proper terms, we have

$$-\frac{r^2}{h^2}\frac{dR}{dr} = \frac{m^2}{2a}\{-\tfrac{1}{2} - \tfrac{3}{2}\cos 2(\theta - \theta') + \tfrac{3}{2}e\cos(\theta - \omega) + \tfrac{9}{4}e\cos(\theta - 2\theta' + \omega) - \tfrac{3}{2}e'\cos(\theta' - \omega')\}.$$

19. Lastly, to find $\frac{1}{h^2}\frac{dr}{d\theta}\frac{dR}{d\theta}$, we have, by (6) and (7), proceeding exactly as before,

$$\frac{1}{h^2}\frac{dr}{d\theta}\frac{dR}{d\theta} = -\frac{3m'r^2}{a\mu r'^3}\frac{dr}{d\theta}\sin 2(\theta - \theta')$$

$$= -\frac{3m^2}{2a}e\sin(\theta - \omega)\sin 2(\theta - \theta')$$

$$= -\frac{3m^2}{4a}e\cos(\theta - 2\theta' + \omega).$$

20. Let us now, for brevity, put

$$\left.\begin{aligned} 2(\theta - \theta') = \phi,\quad \theta - 2\theta' + \omega = \psi,\quad \theta' - \omega' = \chi \\ \theta' \text{ here being simply } m\theta + \beta; \end{aligned}\right\} \ldots. (9).$$

Also, for $e\cos(\theta-\omega)$ put $au-1$; and then we have

$$\frac{\mu}{h^2}=\frac{1+e^2}{a}+\frac{m^2}{a}\left\{-\frac{3}{2}\cos\phi+6e\cos\psi\right\}$$

$$-\frac{r^2}{h^2}\frac{dR}{d\theta}=\frac{3m^2}{2}u+\frac{m^2}{a}\left\{-2-\frac{3}{2}\cos\phi+\frac{9}{4}e\cos\psi-\frac{3}{2}e'\cos\chi\right\}$$

$$\frac{1}{h^2}\frac{dr}{d\theta}\frac{dR}{d\theta}=\frac{m^2}{a}\left\{-\frac{3}{4}e\cos\psi\right\}.$$

Hence, (3) becomes

$$\frac{d^2u}{d\theta^2}+\left(1-\frac{3m^2}{2}\right)u$$

$$=\frac{1}{a}\left(1+e^2-2m^2-3m^2\cos\phi+\tfrac{15}{2}m^2e\cos\psi-\tfrac{3}{2}m^2e'\cos\chi.\right)$$

Assume now $\left(\text{putting } 1-\frac{3m^2}{2}=c^2\right)$

$$u=\frac{1}{a}\left\{e\cos(c\theta-\omega)+A+B\cos\phi+C\cos\psi+D\cos\omega\right\};$$

and therefore, by (9),

$$\frac{d^2u}{d\theta^2}+c^2u=\frac{1}{a}\left[c^2A+\left\{c^2-(2-2m)^2\right\}B\cos\phi\right.$$
$$\left.+\left\{c^2-(1-2m)^2\right\}C\cos\psi+(c^2-m^2)D\cos\chi\right].$$

Hence, equating coefficients, neglecting powers and products of m and e above the second order, and putting $\frac{1}{a}(1+e^2-2m^2)=u_1$, we have

$$u=u_1\left\{1+e\cos(c\theta-\omega)+m^2\cos\phi+\frac{15me}{8}\cos\psi-\frac{3m^2e'}{2}\cos\chi\right\}.$$

21. To find t in terms of θ, we have

$$\frac{dt}{d\theta}=\frac{r^2}{h}\quad\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

in which we must substitute the values of h and r already found in Arts. (17) and (20), and then integrate. But since h is found by an integration with respect to θ, and t by another integration with respect to θ, it is *not* sufficient to retain merely the terms of the third order in which the coefficient of θ is small in the expression for $\frac{d(h^2)}{d\theta}$, but we must also preserve all similar terms of the *fourth* order.*

* In Airy's Tracts and Pratt's Mechanical Philosophy the terms of the third order *only* are retained, and therefore it appears to me that t is found by an inconclusive method, so far as terms of the second order are concerned.

To obtain these terms in the expression for $\frac{d(h^2)}{d\theta}$, we must substitute for r^4, r'^3 and θ' their values to the *second* order, namely

$$r^4 = a^4\left\{1 + e^2 - 4e\cos(\theta-\omega) + 5e^2\cos 2(\theta-\omega) - 4m^2\cos\phi - \frac{15me}{2}\cos\psi\right\},$$

$$r'^3 = a'^3\left\{1 - 3e'\cos(\theta' - \omega)\right\};$$

(we may look upon e'^2 as of the same order as m^3).

$$\theta' = m\theta + \beta + 2e'\sin(m\theta + \beta - \omega') - 2me\sin(\theta - \omega).$$

Now it is easy to see that, on account of the expression for $\frac{d(h^2)}{d\theta}$ being multiplied by $\sin 2(\theta - \theta')$, the expansions of r' and θ' will give rise to no terms in which the coefficient of θ is small; we may therefore suppose $r' = a'$ and $\theta' = m\theta + \beta$ simply. But in the expansion of r^4, the term $5e^2\cos 2(\theta - \omega)$ combined with $\sin 2(\theta - \theta')$ gives a term to be retained, namely $-\frac{5}{2}e^2\sin 2(\theta' - \omega)$. We have therefore

$$\frac{d(h^2)}{d\theta} = -3a\mu m^2\left\{\ldots\ldots - \tfrac{5}{2}e^2\sin 2(\theta' - \omega)\right\},$$

$$\therefore\quad h^2 = a\mu\left\{\ldots\ldots - \tfrac{15}{4}me^2\cos 2(\theta' - \omega)\right\}$$

$$\frac{\mu}{h^2} = \frac{1}{a}\left\{\ldots\ldots + \tfrac{15}{4}me^2\cos 2(\theta' - \omega)\right\}.$$

Introducing this additional term in Art. (20), we find

$$u = u_1\left\{\ldots\ldots \tfrac{15}{4}me^2\cos 2(\theta' - \omega)\right\}.$$

22. We may now proceed to find t.

Represent the values of u and h^2 by $u_1(1 + U)$ and $a\mu(1 - e^2)(1 + H)$; then we have, by (10),

$$\frac{dt}{d\theta} = u^{-2}(h^2)^{-\frac{1}{2}} = \frac{1}{u_1^2\sqrt{\{a\mu(1-e^2)\}}}\cdot(1 - 2U + 3U^2 - \tfrac{1}{2}H)\ldots(11).$$

The values of U and H (neglecting in H the term of the third order in which the coefficient of θ is not small) are

$$U = e\cos(c\theta - \omega) + m^2\cos\phi + \frac{15me}{8}\cos\psi - \frac{3m^2e'}{2}\cos\chi + \frac{15me^2}{4}\cos 2(\theta' - \omega),$$

$$H = \frac{3m^2}{2}\cos\phi - \frac{15me^2}{4}\cos 2(\theta' - \theta);$$

U^2 evidently contains one term of the second order and two of the third, namely,

$$e^2 \cos^2 (c\theta - \omega), \quad 2m^2 e \cos (c\theta - \omega) \cos \phi,$$

$$\text{and} \quad \frac{15me^2}{4} \cos (c\theta - \omega) \cos \psi.$$

The second of these gives no term to be retained, the third gives (putting $c = 1$ and observing that $\psi = \theta - 2\theta' + \omega$), $\frac{15me^2}{8} \cos 2(\theta' - \omega)$, and the first is equal to

$$\tfrac{1}{2} e^2 \{1. + \cos 2(c\theta - \omega)\}.$$

Hence the coefficient of the term $15m^2 e \cos 2(\theta' - \theta)$ in (11) is $-\frac{2}{4} + \frac{3}{8} + \frac{1}{8}$, which $= 0$; this term therefore disappears. Therefore, collecting the terms of (11), and putting all the constant part $= \frac{1}{n}$, we have

$$\frac{ndt}{d\theta} = 1 - 2e \cos (c\theta - \omega)$$

$$+ \frac{3e^2}{2} \cos 2(c\theta - \omega) - \frac{11m^2}{4} \cos \phi - \frac{15me}{4} \cos \psi + 3m^2 e' \cos \chi ;$$

and therefore, integrating,

$$nt + \text{const.} = \theta - 2e \sin (c\theta - \omega)$$

$$+ \frac{3e^2}{4} \sin 2(c\theta - \omega) - \frac{11m^2}{8} \sin \phi - \frac{15me}{4} \sin \psi + 3me' \sin \omega.$$

23. We now proceed to shew that these results are not affected by the inclination of the moon's orbit to the plane of the ecliptic.

By Art. (14) p is the only part of R which contains ι; therefore, since, by Art. (16),

$$R = \frac{m'}{r'} + \frac{m' r^2}{r'^3} \frac{3p^2 - 1}{2},$$

we have

$$\frac{dR}{d\iota} = \frac{3m' r^2}{r'^3} p \frac{dp}{d\iota} \ldots\ldots\ldots\ldots\ldots (12),$$

and, by Article (14), observing that $\lambda' = 0$, $\theta'_1 = \theta'$,

$$p = \cos \lambda \cos (\theta_1 - \theta'),$$

$$\therefore \frac{dp}{d\iota} = - \sin \lambda \frac{d\lambda}{d\iota} \cos (\theta_1 - \theta') - \cos \lambda \frac{d\theta_1}{d\iota} \sin (\theta_1 - \theta') \ldots (13).$$

Now, by Article (14),

$$\sin \lambda = \sin \iota \sin (\theta - \nu) \qquad \tan (\theta_1 - \nu) = \cos \iota \tan (\theta - \nu),$$

hence, differentiating and then neglecting higher powers of λ and ι, and putting $\theta_1 = \theta$, we find

$$\sin\lambda\frac{d\lambda}{d\iota} = \iota\sin^2(\theta-\nu), \quad \cos\lambda\frac{d\theta_1}{d\iota} = -\iota\sin(\theta-\nu)\cos(\theta-\nu),$$

therefore, by (13) and (12),

$$\frac{dp}{d\iota} = -\iota\sin(\theta-\nu)\{\sin(\theta-\nu)\cos(\theta-\theta') - \cos(\theta-\nu)\sin(\theta-\theta')\},$$

$$p\frac{dp}{d\iota} = -\iota\sin(\theta-\nu)\sin(\theta'-\nu)\cos(\theta-\theta'),$$

$$\frac{dR}{d\iota} = -\frac{3m'r^2}{r'^3}\iota\cos(\theta-\theta')\sin(\theta-\nu)\sin(\theta'-\nu) \ldots\ldots (14).$$

Hence, if we denote by R_1 the part of R independent of the inclination, we find, integrating,

$$R = R_1 - \frac{3m'r^2}{2r'^3}\iota^2\cos(\theta-\theta')\sin(\theta-\nu)\sin(\theta'-\nu);$$

from which it appears that the part of R depending on the inclination is of the *fourth* order; therefore the only part of the preceding investigation that can be affected by the inclination is the determination of h^2 from $\frac{d(h^2)}{d\theta}$, Article (17), and we need only look out for those terms in which the coefficient of θ is small in $\frac{dR}{d\theta}$. Now

$$\frac{dR}{d\theta} = \frac{dR_1}{d\theta} - \frac{3m'r^2}{2r'^3}\iota^2\{\cos(\theta-\theta')\cos(\theta-\nu) - \sin(\theta-\theta')\sin(\theta-\nu)\}\cos(\theta'-\nu)$$

$$= \frac{dR_1}{d\theta} - \frac{3m'r^2}{2r'^3}\iota^2.\cos(2\theta-\theta'-\nu)\sin(\theta'-\nu).$$

The part multiplied by ι^2 here evidently gives no term in which the coefficient of θ is small, we have therefore

$$\frac{dR}{d\theta} = \frac{dR_1}{d\theta};$$

and therefore the value of h^2, found in Article (17), is the correct value when the inclination is taken into account.

Again, in finding the time, instead of $h = r^2\frac{d\theta}{dt}$, we have, by Article (12),

$$h = r^2\left(\frac{d\theta}{dt} - 2\sin^2\frac{\iota}{2}\frac{d\nu}{dt}\right)$$

$$= r^2\left(\frac{d\theta}{dt} - \frac{\iota}{2h}\frac{dR}{d\iota}\right) \text{ by Article (12).}$$

The part involving ι here is of the fourth order, and is therefore to be neglected altogether.

Hence it appears that the supposition that the inclination is zero does not in any way affect the process of finding the values of u and t in terms of θ to the second order of small quantities. It is important to remark that θ is the longitude of the moon measured on the plane of her orbit; and to introduce the longitude measured on the plane of the ecliptic, namely θ_1, we must apply the common *reduction*

$$\theta_1 - \theta = -\frac{\iota^2}{4} \sin 2(\theta - \nu).$$

This explains the reason why the only term in the expression for the time in Airy's Tracts, which depends on the inclination, is the *reduction* just put down.

24. We now proceed to determine the motion of the plane of the moon's orbit.

By Article (12)* we have

$$\frac{d\nu}{d\theta} = \frac{d\nu}{dt}\frac{r^2}{h} = \frac{r^2}{h^2\iota}\frac{dR}{d\iota} \quad \text{and} \quad \frac{d\iota}{dt} = \iota \cot(\theta - \nu)\frac{d\nu}{dt};$$

therefore, by Article (23), equation (12), we have

$$\frac{d\nu}{d\theta} = -3m^2 \cos(\theta - \theta') \sin(\theta - \nu) \sin(\theta' - \nu),$$

$$\frac{d\iota}{d\theta} = -3m^2\iota \sin(\theta - \theta') \sin(\theta - \nu) \sin(\theta' - \nu).$$

In both these put

$$2 \sin(\theta - \nu) \sin(\theta' - \nu) = \cos(\theta - \theta') - \cos(\theta + \theta' - 2\nu),$$

retain only the terms which rise in integration, and the constant term in $\frac{d\nu}{d\theta}$, and we find

$$\frac{d\nu}{d\theta} = -\frac{3m^2}{4}\{1 - \cos^2(\theta' - \nu)\},$$

$$\frac{d\iota}{d\theta} = -\frac{3m^2}{4}\iota \sin 2(\theta' - \nu),$$

which give $\nu = \text{const.} - \frac{3m^2}{4}\theta + \frac{3m}{8}\sin 2(\theta' - \nu)$,

$$\iota = \text{const.} + \frac{3m^2}{8}\cos 2(\theta' - \nu);$$

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* In Arts. (12), (8) and (9), in the expressions for $\frac{dv}{dt}$, $\sin\iota$ is put by mistake for $\tan\iota$.

and these values, substituted in the equation

$$\sin \lambda = \sin \iota \sin (\theta - \nu),$$

give the moon's latitude.

25. We shall conclude by pointing out briefly how the parallactic inequality and the more correct values of the mean motions of the perigee and node are obtained.

In Art. (16) the term of R, involving the third power of z or $\frac{r}{r'}$, is

$$\frac{m'r^3}{r'^4} \cdot \frac{5p^3 - 3p}{2}.$$

In this term neglect the eccentricities altogether, put for $\frac{5p^3 - 3p}{2}$ its value $\frac{5}{8}\cos 3(\theta - \theta') + \frac{3}{8}\cos(\theta - \theta')$, retain only the part multiplied by $\cos(\theta - \theta')$, which rises in integration, and, so far as this term of R is concerned, we find

$$\frac{dR}{dr} = \frac{3m'r^2}{r'^4}\left\{\frac{3}{8}\cos(\theta - \theta')\right\},$$

$$\frac{dR}{d\theta} = \frac{m'r^3}{r'^4}\left\{-\frac{3}{8}\sin(\theta - \theta')\right\};$$

$$\text{and } \therefore \quad -\frac{r^2}{h^2}\frac{dR}{dr} = -\frac{9}{8}\frac{m^2}{a}\cdot\frac{a}{a'}\cos(\theta - \theta') \ldots\ldots (A),$$

$$2r^2\frac{dR}{d\theta} = -\frac{3}{4}a\mu m^2 \cdot \frac{a}{a'}\sin(\theta - \theta');$$

hence in $\frac{\mu}{h^2}$ there arises the term

$$-\frac{3}{4}\frac{m^2}{a}\cdot\frac{a}{a'}\cos(\theta - \theta') \ldots\ldots\ldots (B);$$

and, observing that e is zero and therefore $\frac{dr}{d\theta} = 0$, in the equation (3) there arises the term

$$(A) + (B) \quad \text{or} \quad -\frac{15}{8}\frac{m^2}{a}\cdot\frac{a}{a'}\cos(\theta - \theta').$$

If, therefore, $F\cos(\theta - \theta')$ be the corresponding term in u, we have

$$F\{c^2 - (1-m)^2\} = -\frac{15}{8}\frac{m^2}{a}\frac{a}{a'},$$

and therefore

$$F = -\frac{15}{16}\frac{m}{a}\cdot\frac{a}{a'}.$$

Hence we have

$$u = u_1 \left\{ \ldots\ldots - \frac{15}{16} m \frac{a}{a'} \cos(\theta - \theta') \right\}.$$

The term here obtained is the parallactic inequality. (See Airy's *Tracts*.)

26. To obtain the more correct value of the mean motion of the perigee, we must look out in the second member of equation (3) for terms of the form $Ae \cos(\theta - \omega)$. Now it is easy to see that, when we substitute for r, r' and θ' their values to the second order (see Art. 21), in order to expand the second member of (3) to the fourth order, terms of the form $Ae \cos(\theta - \omega)$ will arise only from the combination of the quantities me cos (or sin) $(\theta - 2\theta' + \omega)$ and cos (or sin) $2(\theta - \theta')$. Hence, confining our attention only to the term $me \cos(\theta - 2\theta' + \omega)$, *i.e.* putting for r

$$a \left\{ 1 - \frac{15me}{8} \cos(\theta - 2\theta' + \omega) \right\},$$

and for r' and θ', a and $m\theta + \beta$ simply, we find (see Art. 17)

$$\frac{d(h^2)}{d\theta} = -3a\mu m^2 \left\{ 1 - 4 . \frac{15me}{8} \cos(\theta - 2\theta' + \omega) \right\} \sin 2(\theta - \theta')$$

$$= -3a\mu m^2 \left\{ -\frac{15me}{4} \sin(\theta - \omega) \ldots\ldots \right\},$$

$$h^2 = a\mu \left\{ \ldots\ldots - \frac{3.15}{4} m^2 e \cos(\theta - \omega) . \ldots \right\};$$

hence, and putting $au - 1$ for $e \cos(\theta - \omega)$, we find

$$\frac{\mu}{h^2} = \ldots\ldots + \frac{3.15}{4} m^2 u \ldots\ldots \quad (A).$$

Again, we have (see Art. 18)

$$-\frac{r^2}{h^2} \frac{dR}{dr} = -\frac{m^2}{a} \left\{ 1 - 3 . \frac{15me}{8} \cos(\theta - 2\theta' + \omega) \right\} \frac{3 \cos 2(\theta - \theta')}{2}$$

$$= \frac{m^2}{a} \left\{ \ldots\ldots \frac{9.15}{32} me \cos(\theta - \omega) . \ldots\ldots \right\}$$

$$= \ldots\ldots\ldots \frac{9.15}{32} m^2 u \ldots\ldots\ldots \quad (B).$$

Lastly, we have (see Art. 19)

$$\frac{dr}{d\theta} = a \left\{ \frac{15me}{8} \sin(\theta - 2\theta' + \omega) \right\},$$

and $\therefore \quad \frac{1}{h^2}\frac{dr}{d\theta}\frac{dR}{d\theta} = -\frac{3m^2}{2a}\cdot\frac{15me}{8}\sin(\theta - 2\theta' + \omega)\sin 2(\theta - \theta')$

$$= \ldots\ldots - \frac{3.15}{32}\cdot\frac{m^3e}{a}\cos(\theta - \omega)\ldots\ldots$$

$$= \ldots\ldots - \frac{3.15}{32}m^3u \ldots\ldots\ldots\ldots\ldots\ldots (C).$$

Hence, adding (A), (B) and (C), we have a term

$$\frac{3.15}{4}\left\{1 + \frac{3}{8} - \frac{1}{8}\right\}m^3u = \frac{225}{16}m^3u.$$

Hence the differential equation for u (see Art. 20) becomes

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{3m^2}{2}\right)u = \ldots\ldots \frac{225}{16}m^3u\ldots\ldots$$

$$\text{or} \quad \frac{d^2u}{d\theta^2} + \left(1 - \frac{3m^2}{2} - \frac{225}{16}m^3\right)u = \ldots\ldots\ldots$$

Hence we have $\quad c^2 = 1 - \frac{3m^2}{2} - \frac{225}{16}m^3,$

$$c = 1 - \frac{3m^2}{4} - \frac{225}{32}m^3,$$

which determines the more accurate value of the mean motion of the perigee. (See Airy's *Tracts*.)

27. The more accurate value of the mean motion of the node is easily obtained from the equation (see Art. 24)

$$\frac{d\nu}{d\theta} = -\frac{3m^2}{4}\{1 - \cos 2(\theta' - \nu)\}$$

by putting for ν, in the second member, its first approximate value

$$\nu = \nu_1 + \frac{3m}{8}\sin 2(\theta' - \nu_1),$$

which, expanding $\cos 2(\theta' - \nu)$, and retaining only the constant part, gives

$$\frac{d\nu}{d\theta} = -\frac{3m^2}{4}\left\{1 - \cos 2(\theta' - \nu_1) - \frac{3m}{4}\sin^2 2(\theta' - \nu_1)\right\}$$

$$= -\frac{3m^2}{4} + \frac{9m^3}{32},$$

which determines the more accurate value of the mean motion of the node. (See Airy's *Tracts*.)

M. O. B.

II.—ON THE EQUATION $(D + a)^n\ y = X$.

THIS short paper is intended to illustrate the formation of all the differential equations connected with an equation of the n^{th} order, namely, its successive primitives of the $(n-1)^{\text{th}}$, $(n-2)^{\text{th}}$, &c. orders. One of the greatest encouragements in the use of the calculus of operations is the perfect view which it gives of all the primitives of a linear equation of the form $(D-a)(D-b)\ldots\ldots y = X$, allowing any one of them to be singled out and formed independently of the rest. This is not so easily done when two or more of the set a, b, &c. are equal, the reason being that some of the primitives in such a case are not linear *with constant coefficients*. I do not intend to consider all the combinations of primitives which may occur in an equation of the general form $(D+a)^n\ (D+b)^p \ldots\ldots$ $y = X$, but only the particular form $(D+a)^n\ y = X$.

To take first the simplest form, let $a = 0$, or $y^{(n)} = X$. The discussion of this easy case at length will allow of brevity in treating the more general case. First, it is desirable to express $\int dx)^n \phi x$ in terms of $\int x^n \phi x dx$: let this last be P_n, we have then the following theorems:

$$\Gamma(n+1) \int dx)^{n+1} \phi x = x^n P_0 - n x^{n-1} P_1 + \&c.;$$

$$\int x^m P_n dx = \frac{x^{m+1} P_n - P_{m+n+1}}{m+1},$$

$$\int ABdx = A \int dx)\ B - A' \int dx)^2\ B + A'' \int dx)^3\ B - \&c.$$

From the last theorem (which is an obvious extension of John Bernoulli's) it follows that if A be a rational and integral function of the n^{th} degree, $\int ABdx$ is always integrable when B can be integrated $n+1$ times. Hence $\rho x \,.\, y^{(n)}$ is integrable *per se*, if $\rho x = a_0 + a_1 x + \ldots\ldots + a_{n-1} x^{n-1}$: its integral being $\rho x \,.\, y^{(n-1)} - \rho' x \,.\, y^{(n-2)} + \&c.$ This must be called a known theorem;* but it certainly is not well known.

The mode of proceeding with $y^{(n)} = X$ is now obvious enough: let our instance be $y''' = X$, and let $\int X x^n dx = P_n$. Multiply successively by x^0, x^1, x^2, x^3, and integrate; we thus get the four primitives of the third degree, which are (the constants being included in the symbols P_0 &c.)

(1) $y''' = P_0$,

(2) $xy''' - y'' = P_1$,

(3) $x^2 y''' - 2xy'' + 2y' = P_2$,

(4) $x^3 y''' - 3x^2 y'' + 6xy' - 6y = P_3$.

* It is a particular case, deducible from the general method of finding integrating factors for linear expressions.

To find the six primitives of the second order, observe that the first side of (1) is integrable, and remains so when multiplied by x or x^2; also that the first side of (2), or the same multiplied by x, is integrable; also that the first side of (3) is integrable. Here are then the processes for finding the six equations required, which are

$$(5)\quad y'' = xP_0 - P_1,$$
$$(6)\quad xy'' - y' = \tfrac{1}{2}(x^2P_0 - P_2),$$
$$(7)\quad x^2y'' - 2xy' + 2y = \tfrac{1}{3}(x^3P_0 - P_3),$$
$$(8)\quad xy'' - 2y' = xP_1 - P_2,$$
$$(9)\quad x^2y'' - 3xy' + 3y = \tfrac{1}{2}(x^2P_1 - P_3),$$
$$(10)\quad x^2y'' - 4xy' + 6y = xP_2 - P_3.$$

To find the four primitives of the first order, observe that (5) is integrable, and remains so when multiplied by x, and that (6) and (8) are integrable. The four equations required are

$$(11)\quad y' = \tfrac{1}{2}(x^2P_0 - 2xP_1 + P_2),$$
$$(12)\quad xy' - y = \tfrac{1}{3}x^3P_0 - \tfrac{1}{2}x^2P_1 + \tfrac{1}{6}P_3,$$
$$(13)\quad xy' - 2y = \tfrac{1}{6}x^3P_0 - \tfrac{1}{2}xP_2 + \tfrac{1}{3}P_3,$$
$$(14)\quad xy' - 3y = \tfrac{1}{2}x^2P_1 - xP_2 + \tfrac{1}{2}P_3.$$

Finally, integration of (11) gives the primitive of *the order zero*,

$$(15)\quad y = \frac{1}{2.3}(x^3P_0 - 3x^2P_1 + 3xP_2 - P_3).$$

To extend this to the case of $(D+a)^n y = X$. Search in the common way for the factor which renders $(D+a)^n y$ integrable *per se*; it will be found to be $\varepsilon^{ax}(c_0 + c_1x + \ldots + c_{n-1}x^{n-1})$, say $\varepsilon^{ax}.\rho x$. To find $D^{-1}.\,\varepsilon^{ax}\rho x\,(D+a)^n\,y$, transform it into $D^{-1}\varepsilon^{ax}\rho x\,\varepsilon^{-ax}D^n\varepsilon^{ax}y$ or $D^{-1}\,(\rho x D^n \varepsilon^{ax}y)$, which, on account of the degree of ρx, appears in finite form thus,

$$(\rho x D^{n-1} - \rho' x D^{n-2} + \ldots \pm \rho^{(n-1)} x D^0)\,\varepsilon^{ax}y$$
$$= \varepsilon^{ax}\{\rho x\,(D+a)^{n-1} - \rho' x\,(D+a)^{n-2} + \ldots \pm \rho^{(n-1)} x\,(D+a)^0\}\,y.$$

Proceed thus with $(D+a)^n\,y = X$. Let $\int \varepsilon^{ax} x^n \phi x dx$ be signified by P_n. We have then, as before,

$$\Gamma(n+1)(D+a)^{-(n+1)}\phi x = \varepsilon^{-ax}\left(x^nP_0 - nx^{n-1}P_1 + n\frac{n-1}{2}x^{n-2}P_2 - \ldots\right)$$

$$\int x^m P_n dx = \frac{x^{m+1}P_n - P_{m+n+1}}{m+1}$$

$$(D+a)^{-1}.AB = A(D+a)^{-1}B - A'(D+a)^{-2}B + A''(D+a)^{-3}B - \ldots$$

Though it may be worth while to exhibit these theorems in terms of the operation $D+a$, yet it is not necessary to pro-

ceed further; for $(D+a)^n\, y = X$ is $D^n(\varepsilon^{ax}y) = \varepsilon^{ax}X$, and the rule of proceeding is to solve $y^{(n)} = \varepsilon^{ax}X$, as in the manner above given, and then to write in the several equations $\varepsilon^{ax}y$, $\varepsilon^{ax}(D+a)\,y$, $\varepsilon^{ax}(D+a)^2y$, instead of y, y', y'', &c. Thus, $xy'' - 2y' = x\int X dx - \int Xx^2dx$ being one of the primitives of $y''' = X$, it follows that

$$x\varepsilon^{ax}(D+a)^2\, y - 2\varepsilon^{ax}(D+a)\, y = x\int X\varepsilon^{ax}x dx - \int X\varepsilon^{ax}x^2dx,$$

or $$xy''+(2ax-2)\,y' + (a^2x-2a)\,y = \varepsilon^{-ax}x\int X\varepsilon^{ax}x dx - \varepsilon^{-ax}\int X\varepsilon^{ax}x^2dx,$$

is one of the primitives of

$$y^{iv} + 4ay''' + 6a^2y'' + 4a^3y' + a^4y = X.$$

The following theorem corresponds to that of integration by parts $(D+a)^{-1}\{A\,(D+a)\,B\} = AB - (D+a)^{-1}(BDA)$, which is an immediate consequence of the simple symbolical relation

$$\frac{D}{D+a} = 1 - \frac{a}{D+a}.$$

A. D. M.

III.—ELEMENTARY DEMONSTRATION OF DUPIN'S THEOREM.

If there be three series of surfaces, such that all the surfaces of each series cut the surfaces of the other two series at right angles, the theorem to be proved is that the lines of intersection of any one of the surfaces of the three series, with the surfaces of the two conjugate series, are its lines of curvature.

Let O be any point in which three conjugate surfaces intersect, and let the rectangular axes OX, OY, OZ be perpendicular to the tangent planes of the three surfaces at O. Let

$$f(x, y, z) = \lambda \ldots\ldots\ldots\ldots (a)$$
$$f_1(x, y, z) = \lambda_1 \ldots\ldots\ldots\ldots (a_1)$$
$$f_2(x, y, z) = \lambda_2 \ldots\ldots\ldots\ldots (a_2)$$

be the equations of the three series; and, when proper values are attached to $\lambda, \lambda_1, \lambda_2$, let (a) be the surface touched by YOZ, (a_1) by ZOX, and (a_2) by XOY.

Hence, when $x = 0$, $y = 0$, $z = 0$, we have

$$\left.\begin{aligned} \frac{df}{dy} &= 0, \quad \frac{df}{dz} = 0 \\ \frac{df_1}{dz} &= 0, \quad \frac{df_1}{dx} = 0 \\ \frac{df_2}{dx} &= 0, \quad \frac{df_2}{dy} = 0 \end{aligned}\right\} \ldots\ldots\ldots (b).$$

Now, since the system is orthogonal,

$$\left.\begin{aligned} \frac{df_1}{dx}\frac{df_2}{dx}+\frac{df_1}{dy}\frac{df_2}{dy}+\frac{df_1}{dz}\frac{df_2}{dz}&=0\\ \frac{df_2}{dx}\frac{df}{dx}+\frac{df_2}{dy}\frac{df}{dy}+\frac{df_2}{dz}\frac{df}{dz}&=0\\ \frac{df}{dx}\frac{df_1}{dx}+\frac{df}{dy}\frac{df_1}{dy}+\frac{df}{dz}\frac{df_1}{dz}&=0 \end{aligned}\right\} \ldots\ldots(c).$$

Differentiating the first of these equations with respect to x, the second with respect to y, and the third with respect to z; putting x, y, and z each = 0, in the result, and making use of equations (b), we have

$$\left(\frac{df_1}{dy}\right)\left(\frac{d^2f_2}{dxdy}\right)+\left(\frac{df_2}{dz}\right)\left(\frac{d^2f_1}{dzdx}\right)=0,$$
$$\left(\frac{df}{dx}\right)\left(\frac{d^2f_2}{dxdy}\right)+\left(\frac{df_2}{dz}\right)\left(\frac{d^2f}{dydz}\right)=0,$$
$$\left(\frac{df}{dx}\right)\left(\frac{d^2f_1}{dxdz}\right)+\left(\frac{df_1}{dy}\right)\left(\frac{d^2f}{dydz}\right)=0,$$

the brackets denoting that in the quantities enclosed, x, y, z are put = 0. From these equations we conclude that

$$\left(\frac{d^2f}{dydz}\right),\quad \left(\frac{d^2f_1}{dzdx}\right),\quad \left(\frac{d^2f_2}{dxdy}\right) \text{ are each} = 0.$$

Now, since YOZ is the tangent plane to the surface (a) at O, we have

$$x=\frac{1}{2}\left\{\left(\frac{d^2x}{dy^2}\right)y^2+2\left(\frac{d^2x}{dydz}\right)yz+\left(\frac{d^2x}{dz^2}\right)z^2\right\}+\&c.\ldots.(d)$$

for points in the surface adjacent to O. To determine $\left(\frac{d^2x}{dydz}\right)$, we have, from ($a$),

$$\frac{dx}{dy}=-\frac{\frac{df}{dy}}{\frac{df}{dx}}.$$

Hence, differentiating with regard to z, and putting x, y, and z each = 0, in the result,

$$\left(\frac{d^2x}{dydz}\right)=-\frac{\left(\frac{d^2f}{dydz}\right)}{\left(\frac{df}{dx}\right)}=0.$$

Hence (d) becomes

$$x = \frac{1}{2}\left\{\left(\frac{d^2x}{dy^2}\right)y^2 + \left(\frac{d^2x}{dz^2}\right)z^2\right\} + \&c.$$

Hence the planes of xy and xz contain the principal sections of (a) through O, and therefore the lines of intersection of (a_1) and (a_2) with (a), touch the principal sections of (a) at O. Now O may be any point in one of the surfaces (a), and therefore each of these surfaces has its lines of curvature traced upon it by the surfaces of the series (a_1), (a_2). Similarly it may be shewn that each surface, (a_1) has its lines of curvature traced by (a_2) and (a), and each surface (a_2) by (a) and (a_1), which is the theorem to be proved.

By differentiating the first of equations (c) with regard to y and z respectively, and putting x, y, and z each = 0 in the results, and performing corresponding operations on the second and third, we obtain expressions for

$$\left(\frac{d^2f_1}{dx^2}\right), \left(\frac{d^2f_2}{dx^2}\right), \left(\frac{d^2f}{dy^2}\right), \left(\frac{d^2f_2}{dy^2}\right), \left(\frac{d^2f}{dz^2}\right), \left(\frac{d^2f_1}{dz^2}\right),$$

in terms of $\left(\frac{d^2f}{dx^2}\right), \left(\frac{d^2f_1}{dy^2}\right), \left(\frac{d^2f_2}{dz^2}\right),$

which lead directly to the expression for the curvatures of the principal sections of the three surfaces at O, given by Lamé, in his *Memoir on Curvilinear Co-ordinates*. P. Q. R.

IV.—NOTE ON A DEFINITE MULTIPLE INTEGRAL.

In the XVIII[th] number of this *Journal* Mr. Boole pointed out the incorrectness of a theorem given by M. Catalan. The following pages contain a brief demonstration of the result to which he was led. Both he and M. Catalan made use of what is generally known as Liouville's theorem, and thus perhaps rendered their analysis less simple than it would otherwise have been.

Let us transform the integral

$$\int dx_1 \ldots \int dx_n f(a_1x_1 + \ldots + a_nx_n)$$

by the assumption

$$a_1x_1 + \ldots a_nx_n = au_1 \ldots (a^2 = \Sigma a^2),$$

and by $(n-1)$ other linear relations connecting $x_1 \ldots x_n$ and $u_1 \ldots u_n$, and such that

$$\Sigma x^2 = \Sigma u^2.$$

Then, as is well known, $dx_1 \ldots dx_n$ is to be replaced by $du_1 \ldots du_n$, and thus

$$\int dx_1 \ldots \int dx_n f(a_1x_1 + \ldots + a_nx_n) = \int du_1 f au_1 \int du_2 \ldots \int du_n \ldots (1).$$

Let the integrations on the first side of this equation include all values of the variables which do not transgress the limits

$$\Sigma x^2 = A^2, \quad \Sigma x^2 = B^2,$$

where B is supposed to be greater than A. Then, as $\Sigma x^2 = \Sigma u^2$, the corresponding limits on the second side of the equation are

$$\Sigma u^2 = A^2, \quad \Sigma u^2 = B^2.$$

Transform the integral in x by assuming

$$x_1 = r\cos\theta_1, \quad x_2 = r\sin\theta_1\cos\theta_2 \ldots \quad x_n = r\sin\theta_1 \ldots \sin\theta_{n-1};$$

and that in u, by similar assumptions,

$$u_1 = r\cos\theta'_1 \quad u_2 = r\sin\theta'_1\cos\theta'_2 \ldots u_n = r\sin\theta'_1 \ldots \sin\theta'_{n-1}.$$

I may be allowed to mention that this transformation, which appears to have been given for the first time by Mr. Boole, in the last number of the *Journal*, had occurred to me before I had seen his paper. His analysis leads at once to the conclusion that $dx_1 \ldots dx_n$ is to be replaced by

$$r^{n-1}\sin^{n-2}\theta_1 \ldots \sin\theta_{n-2} \quad dr d\theta_1 \ldots d\theta_{n-1}.$$

This may be also proved by successive substitutions in the manner pointed out in the case of three variables by Mr. A. Smith, in the first volume of the *Journal*.

Thus (1) becomes, since $\Sigma x^2 = \Sigma u^2 = r^2$,

$$\int_A^B r^{n-1}dr \int\sin^{n-2}\theta_1 d\theta_1 \ldots \int d\theta_{n-1} f\{r(a_1\cos\theta_1 + \ldots a_n\sin\theta_1 \ldots \sin\theta_{n-1})\}$$

$$= \int_A^B r^{n-1}dr \int\sin^{n-2}\theta'_1 f(ar\cos\theta'_1)\, d\theta'_1 \int\sin^{n-3}\theta'_2 d\theta'_2 \ldots$$
$$\int\sin\theta'_{n-2} d\theta'_{n-2} \int d'\theta_{n-1}.$$

The limits for θ and θ' are the same.

That this equation may subsist for all values of A and B, it is necessary and sufficient that

$$\int\sin^{n-2}\theta_1 d\theta_1 \ldots \int d\theta_{n-1} f\{r(a_1\cos\theta_1 + \ldots a_n\sin\theta_1 \ldots \sin\theta_{n-1})\}$$
$$= \int\sin^{n-2}\theta'_1 f(ar\cos\theta'_1)\, d\theta'_1 \int\sin^{n-3}\theta'_2 d\theta'_2 \ldots \int d\theta'_{n-1} \ldots\ldots (2).$$

With respect to the limits of θ and θ', it is not difficult to perceive that if $\theta_1 \ldots \theta_{n-1}$ are taken between the limits 0 and π, $x_1 \ldots x_{n-1}$ will receive all the values of which they are capable, namely, all that included between $-r$ and $+r$; and that the same set of values cannot occur more than once. But in order that x_n may vary from 0 to $-r$, it is necessary to extend the superior limit of θ_{n-1} from π to 2π. Thus the limits of θ_{n-1} are 0 and 2π, while those of the other variables $\theta_1 \ldots \theta_{n-2}$ are 0 and π. And similarly for θ'.

On the second side of (2) we have the factor

$$\int_0^{\pi} \sin^{n-3}\theta'_2 d\theta'_2 \int_0^{\pi} \sin^{n-4}\theta'_3 d\theta'_3 \ldots \int_0^{2\pi} d\theta'_{n-1}, \ldots (A).$$

Let n be odd, then we shall have

$$\int_0^{\pi} \sin^{n-3}\theta'_2 d\theta'_2 \int_0^{\pi} \sin^{n-4}\theta'_3 d\theta'_3$$
$$= \frac{(n-4)\ldots 3 \cdot 1}{(n-3)\ldots 4 \cdot 2} \cdot \frac{(n-3)\ldots 4 \cdot 2}{(n-4)\ldots 3 \cdot 1} 2\pi = \frac{2\pi}{n-3}.$$

Similarly

$$\int_0^{\pi} \sin^{n-5}\theta'_4 d\theta'_4 \int_0^{\pi} \sin^{n-6}\theta'_5 d\theta'_5 = \frac{2\pi}{n-5}, \quad \&c. = \&c.$$

Lastly

$$\int_0^{2\pi} d\theta'_{n-1} = 2\pi.$$

Thus $(A) = 2 \dfrac{\pi^{\frac{n-1}{2}}}{\left(\frac{n-1}{2} - 1\right)\left(\frac{n-1}{2} - 2\right)\ldots 2.1} = \dfrac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}.$

Again, if n be even, we get

$$(A) = 2 \frac{\pi^{\frac{n-2}{2}}}{\left(\frac{n-1}{2} - 1\right)\left(\frac{n-1}{2} - 2\right)\ldots \frac{3}{2}\frac{1}{2}}.$$

And this, multiplied by $\dfrac{\sqrt{(\pi)}}{\Gamma(\frac{1}{2})}$, or unity, gives, as before,

$$(A) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)};$$

and thus (2) becomes, making $r = 1$,

$$\int_0^{\pi} \sin^{n-2}\theta_1 d\theta_1 \int_0^{\pi} \sin^{n-3}\theta_2 d\theta_2 \ldots \int_0^{2\pi} d\theta_{n-1} f(a_1 \cos\theta_1 + \ldots \&c.)$$
$$= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^{\pi} \sin^{n-2}\theta'_1 d\theta'_1 f(a \cos\theta'_1). \ldots (3).$$

This is the general result given at the end of Mr. Boole's paper, and which includes the others.

R. L. E.

IV.—NOTE ON SOME POINTS IN THE THEORY OF HEAT.

In problems relative to the motion of heat in solid bodies, the initial distribution, which is entirely arbitrary, is usually one of the data. When this is the case, and the circumstances in which the body is placed are known, the distribution at any subsequent period is fully determined, and if our analysis had sufficient power, would become known in every case. The solution of the problem would be an expression for the temperature of any point in the body in terms of the co-ordinates of the point, and the times measured from the instant at which the distribution is given.

It is in many cases an interesting investigation to examine what this expression becomes when negative values are assigned to the time. If, for a particular negative value $-\tau$, we find that the expression gives an actual arithmetical value for the temperature of every point in the body, the distribution represented will be such that, if it had existed at a time τ before the initial instant, the given initial distribution would have been produced by the spontaneous motion of the heat.

It is clear, however, that the arbitrary initial distribution may be of such a nature that it cannot be the natural result of any previous possible distribution, or that it cannot be any stage except the first in a system of varying temperatures. This, for instance, will be the case if there be any abrupt transitions in the initial temperatures of adjacent points, or if the curves representing the initial temperatures of points situated along any straight line through the body, have cusps or angular points; for, though we may suppose such a distribution to be arbitrarily made, it could obviously not be produced by the spontaneous motion of heat from any other preceding distribution, and all such abrupt transitions or angles which may exist in an initial distribution, will disappear instantaneously after the motion has commenced.

If, however, in any case, a complete solution of the problem has been obtained, that is, if an expression for the temperature of any point, in terms of its co-ordinates, and the time has been found, this expression must assume some *form*, when negative values are given to the time; and it is my object here to examine the nature of this form, when the initial state is such as not to be deducible from a previous distribution.

The simplest case is that of the linear motion of heat, and as from it we are enabled to understand the nature of the

problem in its most general form, I shall, for the present, confine myself to this case. The case of a thin rod, protected from any lateral radiation, or of an infinite solid, in which the temperature is distributed in parallel isothermal planes, is what is usually contemplated in speaking of the linear motion of heat; but the case of a thin rod losing heat by radiation from its sides may also be readily reduced to the same as the two former. For simplicity, however, we may suppose that there is no lateral radiation.

Let a fixed point O in the rod be considered as origin, let the distance to any point P in the rod be x, and let the temperature of P, at the time t from the initial instant, be v. The equation of the motion of the heat will be

$$\frac{dv}{dt} = \frac{d^2v}{dx^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

if, for brevity, we choose the unit of heat such that the conducting power of the body, referred to a unit of its volume, may be unity.

The complete integral of this equation is

$$v = \Sigma A_i \varepsilon^{-m_i^2 t} \cos(m_i x + n_i) \ldots\ldots\ldots\ldots (2).$$

Let ${}_0v$ be the initial temperature of P. Then, putting $t = 0$, in (2), we have

$${}_0v = \Sigma A_i \cos(m_i x + n_i) \ldots\ldots\ldots\ldots (3).$$

Now, whatever be the arbitrary nature of the function ${}_0v$, whether continuous or discontinuous, it has been shewn by Fourier that it may be represented by a series such as the second member of (3), and he has shewn how, when the value of ${}_0v$, corresponding to any value of x, is given, the constants A_i, m_i, n_i may be determined. The series thus found is an actual quantitative representation of ${}_0v$, and is necessarily convergent. In fact, the proof of its convergence in every case must be included in the demonstration of the possibility of representing any function ${}_0v$ by the series (3).

Now the expression for v at time t, is found by multiplying the respective terms of the second members of (3), by the quantities

$$\varepsilon^{-m_1^2 t}, \quad \varepsilon^{-m_2^2 t}, \text{ \&c.} \ldots\ldots\ldots\ldots\ldots (a);$$

a series which converges when t is positive, (m_1, m_2, &c. being arranged in ascending order of magnitude). Hence, when t is positive, the series for v is still more convergent than the series for ${}_0v$, and therefore (2) gives a convergent series for v for every positive value of t. If, however, the time considered be τ *before* the initial instant, the series (a) will become

$$\varepsilon^{m_1^2 \tau}, \quad \varepsilon^{m_2^2 \tau}, \text{ \&c.} \ldots\ldots\ldots\ldots\ldots (b),$$

which is divergent. Hence it will depend on the degree of the convergence of the series for ${}_0v$, whether the expression of v in this case be convergent or divergent. In the latter case the distribution represented will be impossible, and therefore there will be no distribution from which the initial distribution can be derived, by spontaneous motion, in the time τ. There are three cases of the initial distribution, which we must specially consider.

1. If the convergence of the expression for ${}_0v$, or of the series of coefficients A_1, A_2, &c.

be of a *lower order* than that of the series (a); that is, if the ultimate convergence of this series be less than that of (a), for every positive value of t, then for any finite value, however small, of τ, the series

$$A_1\varepsilon^{-m_1{}^2\tau}, \quad A_2\varepsilon^{-m_2{}^2\tau}, \quad \&c.$$

will be ultimately divergent. Hence, in this case, for any time, however small, before the initial instant, the distribution will be impossible, and therefore the given distribution cannot be any stage but the first in a system of varying temperatures.

2. If the convergence of the series for ${}_0v$ be ultimately of the *same order* as that of the series (a); that is, if the coefficients A_1, A_2, &c. can be put under the forms

$$a_1\varepsilon^{-m_1{}^2\tau'}, \quad a_2\varepsilon^{-m_2{}^2\tau'}, \quad \&c.,$$

where a_1, a_2, &c. form a series which ultimately has a convergence of a *lower order* than that of the series (a), according to the definition previously given; then, at a time τ' before the initial instant, the distribution is represented by

$$v' = \Sigma a_i \cos(m_i x + n_i),$$

which belongs to the first class considered, and is therefore an essentially primitive distribution. Hence, in this case, a finite *age*, τ' may be assigned to the given initial distribution.

3. If the convergence of the series for ${}_0v$ be of a *higher order* than that of the series (a); or if, for any finite value of τ, however great, the series

$$A_1\varepsilon^{-m_1{}^2\tau}, \quad A_2\varepsilon^{-m_2{}^2\tau}, \quad \&c.$$

is ultimately convergent; then, for every finite negative value of t, the series for v will converge, and therefore v will have a possible expression. Hence, in this case, no limit can be assigned to the *age* of the initial distribution.

The simplest case of distributions which belong to the third class, is that in which the expression for ${}_0v$ is composed of a finite number of terms of the form

$$A \cos (mx + n);$$

but we may also readily form series of an infinite number of terms, which will be sufficiently convergent to satisfy the condition stated for the third class, and therefore any initial distribution represented by such a series may be deduced from distributions existing previously for any length of time, however great.

These remarks are sufficient to indicate the *nature* of the question proposed. The details of the convergence or divergence of the series employed will depend on the values which must be assigned to m_1, m_2, &c. for satisfying the conditions of any particular problem.

If, instead of the initial distribution of heat in the rod, from $-\infty$ to $+\infty$ being given, we have only a part, that for instance on the positive portion of the rod, we must, to make the problem determinate, have some other condition given. In such cases, the part of the rod, over which the initial distribution is not given may be conceived to be removed, and the second condition will then generally relate to the extremity of the part of the rod considered. In a previous paper (*On the Linear Motion of Heat*, vol. III. p. 170), I have shewn how a solution of the problem may be obtained by determining the initial distribution which must be made on the negative part of the rod, in order that the condition relative to the zero point may be fulfilled. Thus, if the temperature at the zero point, or the extremity of the part of the rod originally given, be constrained to be a given arbitrary function of the time, this constraint may be effected by producing the rod indefinitely in the negative direction, and impressing on the part produced a certain initial distribution, determined by equations (6) and (10) of the Article already referred to, in terms of the arbitrary function of the time, and the given initial distribution on the positive side. In many cases, however, the distribution so determined will be impossible, either over the whole extent of the negative part of the rod, or from some point at a finite distance on the negative side, to an infinite distance. Notwithstanding this impossibility, the solution of the problem obtained by the method described above will still give a finite possible expression for the temperature of any point on the positive side, at any time subsequent to the initial instant, which will be the tempera-

ture that would be actually assumed by the point, if the temperature of the zero point had, by some external application, been constrained to satisfy the given condition; and the only thing indicated by the impossibility of the distribution on the negative side will be that this constraint cannot be effected actually, by adding the negative part of the rod, with a certain initial distribution of heat, to the given positive part. It is unnecessary to enter into this question separately here, as the details are very similar to those given above. It should be remarked however that the possibility or impossibility of the distribution on the negative part will depend entirely on the function of the time, which expresses the variable temperature of the zero point, and not at all on the given initial distribution on the positive part; and on this account, if the initial distribution on the negative part be impossible, all its subsequent forms will generally be impossible also.

Before leaving this subject, it may be well to notice a point relative to the theory of isothermal surfaces, or surfaces of equilibrium, which involves considerations analogous to those with which we have been occupied above.

It is a known theorem that an attraction every where perpendicular to any given closed surface S, may be produced by the distribution of a given quantity of matter m over the surface, according to a law which is in every instance determinate; but in general it will be impossible to produce the same effect by the distribution of matter over any surface in the interior of S, not coinciding with it. If, for instance, S were the surface of a cube, or any surface containing points or edges, this would obviously be impossible, and it would probably also be impossible in the case considered by Poisson, in which S is composed of two spherical surfaces. In every case there will be an infinite series of surfaces of equilibrium without S, becoming ultimately a series of spheres having the centre of gravity of m for their common centre, each of which is such that m exerts an attraction on any point in it in the direction of the normal. Hence, if one of these surfaces, s, be given, we may not only produce an attraction every where perpendicular to it, by matter distributed over itself, but by matter distributed over S, or over any surface between S and s, enclosing the former. Hence s is analogous to a distribution of heat which may be produced by a previously existing distribution. But, unless S itself have the particular property we are considering in s, we cannot produce an attraction perpendicular to s by any distribution on

a surface within S. Thus s is analogous to a distribution of heat which cannot be produced by any previous distribution existing at a time before it greater than a certain limit τ.

Again, in some cases the series of surfaces of equilibrium, of which s is one, may be continued indefinitely inwards, till we arrive at a surface enclosing no space, as for instance when s is an ellipsoid, in which case the surfaces of equilibrium are confocal ellipsoids, and the series within s may be continued till we arrive at an elliptical disc. Such a case is analogous to that of a distribution of heat, which may be produced from a distribution existing an unlimited time before.

P. Q. R.

V.—ON THE INTENSITY OF LIGHT IN THE SHADOW OF A VERY SMALL CIRCULAR DISK.

It has been proved experimentally by M. Arago, that the central point of the shadow of a very small opaque circular disk is sensibly as bright as if the light were not intercepted. Fresnel has given a general explanation of this curious result in the Appendix to his *Memoir on the Undulatory Theory of Light*, (*Mem. de l' Institut.*, tom. v. p. 460. *Ann.* 1821–22), and states that Poisson has proved the same thing by direct integration; but he gives no hint as to where this proof can be found.

The usual method of finding the illumination of a point appears in this instance to present some difficulty. For if (fig. 1) A be the centre of the disk, Q the point in the shadow whose illumination is to be found, and we suppose the waves of light plane and parallel to the plane of the disk, also if $PQ = u$, it is easily seen that the whole vibration at Q will be expressed by

$$2\pi C \int \sin \frac{2\pi}{\lambda} (vt - u) \, . \, du.$$

The superior limit being ∞, and the inferior Qa, a being a point on the rim of the disk.

The integration introduces at the superior limit the rather doubtful expression $\cos \infty$: but, supposing this to vanish, as the nature of the case shews that it ought, for the vibration at Q cannot be affected by the small waves originating at very distant points of the general incident plane wave, still there is another difficulty; for the integration above indicated, if performed, gives

$$C\lambda \cos \frac{2\pi}{\lambda} (vt - Qa)$$

for the total vibration at Q, and therefore the illumination

will be expressed by $C^2\lambda^2$, which expression being independent of the radius of disk, is considered to prove the proposition. But the difficulty is, that this expression is true however large the disk may be, as well as when it is very small, which is manifestly absurd. It is true that the same reasons which led us to neglect the quantity $\cos\alpha$ would teach us to neglect the expression $\cos\frac{2\pi}{\lambda}(vt - Qa)$, when the radius of the disk is very large, and so the illumination would become zero, as it ought: but this does not remove the difficulty from the analysis; for, according to the principle usually adopted, the illumination ought *not* to vanish, when the radius of the disk is considerable; because, since, when a general wave is broken up, the effect due to the small elementary wave is supposed to depend on the distance of the illuminated point from the origin of the small wave, and on that only, it follows that the effect of a small wave spreading from the rim of a disk whose radius is AP (fig. 2) in exciting a vibration at the point Q, will be the same as that of a wave spreading from the rim of a disk whose radius is AP' in exciting a vibration at Q', provided only that $PQ = P'Q'$. In fact the common method does not take into account the effect of *obliquity*, and seems to involve the same error as if, in calculating the illumination of a point by a bright surface according to the principles of common Optics, we were to omit the consideration of the fact that the quantity of light emanating in any direction varies as the sine of the angle of emanation.

This problem has been very elegantly treated in the *Mathematical Journal* (vol. II. p. 141), and the result there obtained is the same as that found above, viz. that the illumination is measured by $C^2\lambda^2$: but the method there given seems to labour under precisely those difficulties which have been just alluded to. The fact seems to be, that the principle of Huyghens, which supposes the illumination of a point on which a luminiferous undulation is incident, to be the same as that which would result from the combination of spherical wavelets diverging with equal intensity in all directions from the elements of the wave-front, cannot be looked upon as a physical principle, but only as an artifice rendered necessary by the state of analysis, and which will not always represent the physical conditions of the problem.

In this paper I have adopted Huyghens' principle so far as this, that I consider the vibration of the ether at a given

point to be the sum of the vibrations which would exist if the wave-front were divided into elements and each of these elements were a centre of disturbance; but I do not suppose the wavelets to diverge spherically, but according to another law which I shall investigate.

With this view I shall first find an integral of the partial differential equation, which seems applicable to the case of light, in the particular case *in which the vibration is symmetrical with respect to a certain axis, and the effect rapidly diminishes as we recede from that axis.*

I shall assume, as the equation applicable to the vibrations of the ether,

$$\frac{d^2\phi}{dt^2} = a^2\left\{\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right\} \ldots\ldots\ldots (1),$$

and let the axis of z be that about which every thing is symmetrical: let $x^2+y^2=\rho$, then ϕ is a function of z, ρ, and t; also

$$\frac{d^2\phi}{dx^2} = 4\,\frac{d^2\phi}{d\rho^2}\,x^2 + 2\,\frac{d\phi}{d\rho},$$

$$\frac{d^2\phi}{dy^2} = 4\,\frac{d^2\phi}{d\rho^2}\,y^2 + 2\,\frac{d\phi}{d\rho};$$

$$\therefore \quad \frac{d^2\phi}{dt^2} = a^2\left\{\frac{d^2\phi}{dz^2} + 4\rho\,\frac{d^2\phi}{d\rho^2} + 4\,\frac{d\phi}{d\rho}\right\} \ldots\ldots (2).$$

Now if ϕ diminishes very rapidly as ρ increases; that is, if we may suppose ρ very small, the term $\rho\,\dfrac{d^2\phi}{d\rho^2}$ will be negligeable as compared with $\dfrac{d\phi}{d\rho}$; or at all events we may integrate upon that hypothesis, and if it should turn out that this is not the case, the method must be abandoned.

Neglecting, then, the term $\rho\,\dfrac{d^2\phi}{d\rho^2}$, our equation becomes

$$\frac{d^2\phi}{dt^2} = a^2\left\{\frac{d^2\phi}{dz^2} + 4\,\frac{d\phi}{d\rho}\right\} \ldots\ldots\ldots (3).$$

Let $$\phi = u \sin\frac{2\pi}{\lambda}(vt - z),$$

where u involves ρ only: differentiating and substituting, we have

$$-\frac{4\pi^2}{\lambda^2}v^2.u = a^2\left\{-\frac{4\pi^2}{\lambda^2}\,u + 4\,\frac{du}{d\rho}\right\},$$

$$\therefore\ u = ae^{-\frac{\pi^2}{\lambda^2}\frac{v^2-a^2}{a^2}\cdot\rho},$$

$$\text{and } \phi = ae^{-\frac{\pi^2}{\lambda^2}\frac{v^2-a^2}{a^2}\rho}\ \sin\frac{2\pi}{\lambda}(vt-z)\ \ldots\ldots(4).$$

Let us now see whether $\rho\dfrac{d^2\phi}{d\rho^2}$ was rightly neglected: $\rho\dfrac{d^2\phi}{d\rho^2} \div \dfrac{d\phi}{d\rho}$ will be small when ρ is small, provided $\dfrac{v^2-a^2}{\lambda^2 a^2}$ is not large: and this is a condition which is undoubtedly fulfilled; for, though λ is very small, a is very large indeed, and v^2 also cannot be very different from a^2. I think, therefore, we are justified in considering that the expression (4) very nearly represents the true physical conditions of the problem: in order that it may do so, however, v must be greater than a, and for this I can give no physical reason; I shall, however, assume that it is so, and proceed to apply formula (4) to the question in hand.

If we take an annulus of the plane incident wave, whose radius is $\sqrt{\rho}$, the area of the annulus will be $\pi d\rho$, and if AQ (fig. 1) $= f$, the vibration at Q due to the waves spreading from the different points of this annulus, will be expressed by

$$\pi a\int_{b^2}^{\infty} e^{-\frac{\pi^2}{\lambda^2}\frac{v^2-a^2}{a^2}\rho}\ \sin\frac{2\pi}{\lambda}(vt-f)\,d\rho$$

$$= \pi a\,\frac{\lambda^2}{\pi^2}\cdot\frac{a^2}{v^2-a^2}\cdot e^{-\frac{\pi^2}{\lambda^2}\frac{v^2-a^2}{a^2}b^2}\ \sin\frac{2\pi}{\lambda}(vt-f),$$

where b = the radius of the disk. And therefore the illumination will be expressed by

$$a^2\,\frac{\lambda^4}{\pi^2}\cdot\left(\frac{a^2}{v^2-a^2}\right)^2 e^{-\frac{2\pi^2}{\lambda^2}\frac{v^2-a^2}{a^2}\cdot b^2}.$$

This expression diminishes very rapidly as b increases, but if b be small it approaches very nearly to the value $a^2\dfrac{\lambda^4}{\pi^2}\left(\dfrac{a^2}{b^2-a^2}\right)^2$, which is its value if $b=0$, or if the light was not intercepted. And this seems to agree sufficiently accurately with the experimental result.

I will now apply the same method to express the illumination at any point within the geometrical shadow.

If we take the projection of the given point on the disk as the origin, and call its distance from the centre g, and

measure θ from the line joining this point and the centre, the equation to the circumference of the disk will be

$$r^2 \sin^2\theta + (r\cos\theta - g)^2 = b^2,$$

or
$$r^2 - 2rg\cos\theta = b^2 - g^2$$

$$r = g\cos\theta + \sqrt{(b^2 - g^2\sin^2 b)}$$

$$\rho = g^2\cos 2\theta + b^2 + 2g\cos\theta\sqrt{(b^2 - g^2\sin^2\theta)}.$$

If instead of $\frac{\pi^2}{\lambda^2}\frac{v^2 - a^2}{a^2}$ we put n, for shortness, and call the illumination at the point in question I, it will be easily seen that

$$\sqrt{I} = \frac{a}{2}\iint e^{-n\rho}d\theta d\rho,$$

where the limits of ρ are the above value and ∞, and the limits of θ are 0 and π, if we double the integral, since only half the disk is included between 0 and π;

$$\therefore \sqrt{I} = \frac{a}{n}\int_0^\pi e^{-n\left[g^2\cos 2\theta + b^2 + 2g\cos\theta\, d\theta\sqrt{(b^2 - g^2\sin^2\theta)}\right]}$$

$$= \frac{a}{n}\, e^{-nb^2}\int_0^\pi e^{-n\left[g^2\cos 2\theta + 2g\cos\theta\, d\theta\sqrt{(b^2 - g^2\sin^2\theta)}\right]}.$$

The integration here indicated cannot be performed generally, but there are some particular cases which are worth noticing.

If g be so small that g^2 and higher powers may be neglected, the integral becomes

$$\int_0^\pi e^{-2nbg\cos\theta}\, d\theta = \int_0^\pi (1 - 2nbg\cos\theta)\, d\theta = \pi,$$

or the illumination is the same as at the centre.

If we take in the square of g, the integral becomes

$$\int_0^\pi e^{-ng^2\cos 2\theta}\, e^{-2nbg\cos\theta}\, d\theta = \int_0^\pi (1 - 2nbg\cos\theta - ng^2\cos 2\theta + 2n^2b^2g^2\cos^2\theta)\, d\theta$$

$$= \pi\,(1 + n^2b^2g^2),$$

$$\therefore \quad I = \frac{\pi^2a^2}{n^2}\, e^{-2nb^2}\,(1 + 2n^2b^2g^2).$$

Hence, the centre is not so bright as an annulus taken very near to it.

There is one other case which I shall consider, viz. for the geometrical shadow, or when $g = b$; in this case the integral becomes

$$\int_0^\pi e^{-nb^2(1+2\cos 2\theta)}\, d\theta = e^{-nb^2}\int_0^\pi e^{-2nb^2\cos 2\theta}\, d\theta,$$

which is less than π, because e^{-nb^2} is < 1, and

$$\int_0^{\pi} e^{-2nb^2 \cos 2\theta}\, d\theta = \int_0^{\pi} (1 - 2nb^2 \cos 2\theta)\, d\theta = \pi, \text{ nearly,}$$

since b is very small. Hence the line of the geometrical shadow will be less bright than the centre. Between these there must be at least one ring of maximum brightness, there may be more.

How far all this agrees with experiment I cannot say, but at least this method of considering the problem appears to embrace the two great facts of the brightness of the centre, and the existence of rings in the shadow.

H. G.

VI.—ON THE MOTION OF THE CENTRE OF GRAVITY OF BROKEN BODIES.

By W. Walton, M.A. Trinity College.

The fundamental principle adopted by Huyghens in the determination of the Centre of Oscillation or Agitation of Complex Pendulums, which forms the subject of the fourth part of the Horologium Oscillatorium, is enunciated in the following words: "Si pendulum e pluribus ponderibus compositum, atque e quiete dimissum, partem quamcunque oscillationis integræ confecerit, atque inde porro intelligantur pondera ejus singula, relicto communi vinculo, celeritates acquisitas sursum convertere, ac quousque possunt ascendere; hoc facto, centrum gravitatis ex omnibus compositæ, ad eandem altitudinem reversum erit, quam ante inceptam oscillationem obtinebat."

From the nature of the argument of this section of the Horologium Oscillatorium, it seems probable, although no where, I believe, expressly stated, and although in opposition to the direct meaning of the words which we have quoted, that Huyghens really supposed each of the disconnected molecules, after arriving at its point of zero velocity, to be there permanently retained; so that the 'centrum gravitatis ex omnibus compositæ' should be considered as permanently assuming its original altitude from the date of the arrival of the molecule of greatest velocity at its maximum altitude. Such a modification of the proposition is at any rate necessary to its truth. The corrected form of the principle which we have mentioned involves indirectly, and in a particular case, the principle of Vis Viva, a generalization which in fact resulted from the principle of Huyghens in the hands of John and Daniel Bernoulli. A demonstration of the principle of Huyghens,

in its unmodified form, has been attempted by Professor Whewell, in a work entitled *The Mechanics of Engineering*, p. 142, by the application of the general principle of Vis Viva. The reasoning however of the Professor is not satisfactory, in consequence of his having omitted to take account of the idea of time, an essential element of the problem.

The object of this paper is to offer a few remarks on the motion of the centre of gravity of the disconnected molecules, supposing them not to be retained in their respective positions of zero velocity, but to be capable of descending spontaneously from their positions of instantaneous rest. It is not necessary to suppose the body to have been oscillating, before the dissolution of the connections of its molecules, about a fixed axis in the manner of a pendulum, but we may imagine it to have descended from rest in any conceivable way; the remarks we are about to make being generally applicable.

Let $m, m', m'' \ldots$ represent the masses of the molecules, $h, h', h'' \ldots$ their initial distances below the horizontal line through the initial position (G) of their centre of gravity; $x, x', x'' \ldots$ their distances below the same line at the moment of the dissolution of their mutual connections; then, if $u, u', u'' \ldots$ denote the velocities of the molecules on the commencement of their independent motions, we have, by the principle of the Conservation of Vis Viva,

$$mu^2 + m'u'^2 + m''u''^2 + \ldots . = 2g\{m(x-h) + m'(x'-h') + m''(x''-h'') + \ldots .\};$$

but, by the property of the centre of gravity of any number of bodies, $mh + m'h' + m''h'' + \ldots . = 0$,

and $mx + m'x' + m''x'' + \ldots . = (m + m' + m'' + \ldots .)\,\bar{x}$,

where $\bar{x}$ denotes the distance of the centre of gravity (H) of the molecules at the moment of their mutual disconnection below the horizontal line through (G); hence

$$mu^2 + m'u'^2 + m''u''^2 + \ldots . = 2g\,(m + m' + m'' + \ldots .)\,\bar{x} \ldots . (1).$$

Again, let $v, v', v'' \ldots$ denote contemporaneous velocities of the molecules in their independent movements, and $y, y', y'' \ldots \ldots$ corresponding contemporaneous altitudes above the horizontal line through (H); also let $k, k', k'' \ldots$ denote the values of $y, y', y'' \ldots$ at the instant of disconnection. We will commence with supposing the molecules to be constrained to move from the moment of detachment along smooth invariable lines, and not to be subject to mutual col-

lision; then, by the theory of the motion of heavy particles, whether the paths of the molecules previous to disconnection be joined continuously with those which they afterwards pursue, or their motions be diverted at the instant of their mutual detachment without loss of velocity into different directions,

$$\left.\begin{aligned} mv^2 &= mu^2 - 2mg\,(y-k),\quad m'v'^2 = m'u'^2 - 2m'g\,(y'-k'),\\ m''v''^2 &= m''u''^2 - 2m''g\,(y''-k''),\ \ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}\ldots\ldots(2);$$

we have therefore

$$mv^2 + m'v'^2 + m''v''^2 + \ldots = mu^2 + m'u'^2 + m''u''^2 + \ldots$$
$$- 2g\,\{m(y-k) + m'(y'-k') + m''(y''-k'') + \ldots\};$$

but, by the property of the centre of gravity of bodies,

$$mk + m'k' + m''k'' + \ldots = 0,$$

and $$my + m'y' + m''y'' + \ldots = (m + m' + m'' + \ldots)\,\bar{y},$$

where $\bar{y}$ denotes the altitude of the centre of gravity of the disconnected molecules at any time above the horizontal line through (H); hence

$$mv^2 + m'v'^2 + m''v''^2 + \ldots$$
$$= mu^2 + m'u'^2 + m''u''^2 + \ldots - 2g\,(m + m' + m'' + \ldots)\,\bar{y} \ldots (3).$$

From this last equation it is evident that $\bar{y}$ can never have a value greater than that given by the equation

$$2g\,(m + m' + m'' + \ldots)\,\bar{y} = mu^2 + m'u'^2 + m''u''^2 + \ldots.$$
$$= 2g\,(m + m' + m'' + \ldots)\,\bar{x} \text{ from (1)};$$

and therefore it is impossible for $\bar{y}$ ever to exceed $\bar{x}$ in magnitude.

It remains to consider whether, or under what circumstances, $\bar{y}$ will ever become equal to $\bar{x}$. Suppose $\bar{y} = \bar{x}$; then, from (3), we see that

$$mv^2 + m'v'^2 + m''v''^2 + \ldots = 0,$$

and therefore, contemporaneously,

$$v = 0,\quad v' = 0,\quad v'' = 0 \ldots.$$

or, from the relations (2),

$$2g\,(y-k) = u^2,\quad 2g\,(y'-k') = u'^2,\quad 2g\,(y''-k'') = u''^2 \ldots;$$

hence we see that, if ever $\bar{y}$ become equal to $\bar{x}$, *every* molecule must have at that instant a zero velocity, or be at the highest point of one extremity of its arc of oscillation. Suppose now that t, τ, denote respectively, the time which m takes to move through the first and second portion of its arc of oscillation, the two portions being supposed to be separated

by the position of m at the instant of its detachment. Let (t', τ'), (t'', τ''), (t''', τ''') represent like times in relation to the molecules m', m'', m'''.... Then, since v, v', v''.... must be contemporaneously equal to zero in order that $\bar{y}$ may be equal to $\bar{x}$, the condition for such equality will evidently be expressed by a class of relations of the form

$$t + n(t + \tau) = t' + n'(t' + \tau') = t'' + n''(t'' + \tau'') = \ldots . (4),$$

where n, n', n''.... are all positive integers. The possibility of satisfying these relations by any finite values of n, n', n''.... as will be evident on a little reflection, will essentially depend upon the forms of the paths described by the molecules subsequently to their disconnection. In case the molecules were to commence their motions at the moment of detachment along cycloidal arcs, starting at the lowest point of each, we should have

$$t = \tau = t' = \tau' = t'' = \tau'' = \ldots .$$

and the relations (4) would become

$$2n + 1 = 2n' + 1 = 2n'' + 1 = \ldots .$$

and therefore

$$n = n' = n'' = \ldots ;$$

or the centre of gravity of the disconnected molecules would rise at each semi-oscillation of each separate molecule into a position on a level with (G). For the sake of another illustration, suppose that the molecules are perfectly elastic and keep rebounding above a horizontal plane in vertical lines; in this case we must put τ, τ', τ''.... all equal to zero, and the relations will become

$$(n + 1)\, t = (n' + 1)\, t' = (n'' + 1)\, t'' = \ldots .$$

or, since

$$u = gt, \quad u' = gt', \quad u'' = gt'' \ldots .$$

$$(n + 1)\, u = (n' + 1)\, u' = (n'' + 1)\, u'' = \ldots ;$$

if, then, u, u', u''.... be quantities relatively commensurable, it is plain that these relations may be satisfied by admissible values of n, n', n''....; on the other hand, if the requisite commensurability do not subsist among the projectile velocities, it is equally evident that the relations cannot be satisfied by any finite values of n, n', n''....; in the one case, then, the centre of gravity of the disconnected molecules will from time to time, at finite intervals, arrive at its original altitude, while in the other case it will never attain to it in any finite time. It is needless to multiply examples of the impossibility of satisfying the formulæ (4) by any finite values of n, n', n''... it being obviously rather a singular coincidence that the requisite commensurability should subsist without previous arrangement than a state of things of probable occurrence.

We have been supposing in the preceding observations that the disconnected molecules oscillate permanently in invariable curves, and that they are free from a liability to mutual collision. We will now suppose the molecules, considered as perfectly elastic, to be capable of assuming any irregular motions whatever, and of impinging against each other. Under this more general consideration of the possible movements, the formula (2) will be no longer applicable, the values of v, v', v''.... obviously depending not only upon u, u', u''.... and the altitudes of ascent, but likewise upon the effects of collision. The formula (3) however, as we know by the application of the Principle of the Conservation of Vis Viva, still continues to hold good; we may therefore shew, as before, that it will be impossible for the centre of gravity of the molecules to rise to its original altitude, unless the velocities v, v', v''.... become simultaneously equal to zero. Let t_1 denote the interval from the instant of detachment to the arrival of the molecule m at its first position of zero velocity; t_2, t_3, t_4.... representing the intervals between its successive arrivals at such positions: let t_1', t_2', t_3'.... denote similar intervals in respect to the molecule m', and so on for the rest. Then, in order that the centre of gravity may attain to its original altitude, we must have

$$t_1 + t_2 + t_3 + \ldots t_n = t_1' + t_2' + t_3' + \ldots + t'_{n'} = t_1'' + t_2'' + t_3'' + \ldots t''_{n''} = \ldots (5),$$

the suffixes n, n', n''.... denoting certain integral numbers.

The probability that for any finite values of n, n', n''.... these indefinitely numerous relations should be verified, is evidently indefinitely small. We may therefore lay it down as a general practical principle, that if a mass of matter, either in its descent from rest or at the instant of its starting, be dismembered into its constituent molecules, which subsequently pursue paths prescribed by the accidental circumstances of gravity or collision, even supposing the molecules to be perfectly elastic and discarding all consideration of friction or atmospheric resistance, its centre of gravity will never again rise to a level with its original position.

The truth of this practical principle, it will be evident, depends not upon any physical impossibility of the arrival of the centre of gravity at its original altitude, but upon the infinite mathematical improbability of the conspiracy of the necessary circumstances within any finite time. The observations which we have made above, in relation to the movements of the disconnected molecules of descending bodies, will

evidently be equally applicable to the case of fluids flowing down from a higher into a lower level, the centre of gravity of which will accordingly oscillate in a lower basin without ever reaching its original altitude.

If instead of supposing the mass to be dismembered into its constituent molecules, we imagine it to be broken merely into fragments of finite magnitude, since the equation (3) will still hold good, we shall still have relations such as those of (5) for the conditions of the rise of the centre of gravity to its original height; it is important to remark, however, that the number of these relations will, according to the new hypothesis, be finite; for if three molecules of each fragment, which are not in a single straight line, ever assume zero velocities, the whole of each fragment will be in a position of instantaneous rest. Thus, if there be p fragments, there will be only $3p-1$ relations of condition; if the mass be supposed never to have fallen to pieces at all, then, according to the same reasoning, we see that there will be only two relations of condition. Thus we come to this general conclusion, that if a mass of matter in descending from any position fall to pieces, the improbability of its centre of gravity ever rising to its original altitude will be the greater, the greater be the number of the fragments; and that, when the fragments degenerate into molecules, the improbability will amount to a practical impossibility. It may perhaps not be superfluous to observe that, if all the particles of a descending body be constrained to move in parallel planes, the whole body will be at rest if any two molecules, not in the same horizontal line, be at rest, and therefore the equations of condition will be reduced to one. If the body oscillate about a fixed axis as a complex pendulum, then if any one molecule be at rest the whole body will be for a moment without motion; and therefore there will be no relations of condition, or the centre of gravity will undoubtedly rise at the end of each oscillation to its original altitude.

VII.—ON THE INVERSE CALCULUS OF DEFINITE INTEGRALS.

By George Boole.

This paper must be considered as the continuation of a paper on the *Transformation of Definite Integrals*, (*Journal*, No. xvii. p. 216.)

M. Liouville has given the following theorem as the basis of an inverse calculus of definite integrals, viz.

$$\int_0^\infty dx\, x^{n-1}\phi(x+a) = (-)^n\,\Gamma(n)\left(\frac{d}{da}\right)^{-n}\phi(a). \ldots (28),$$

a demonstration of which will be found in an able article in this *Journal*, vol. I. p. 113. It is there observed, that if the form of $\phi(x)$ be such as to render the first side infinite, the second may be made to agree with it by the aid of the complementary function, so that the theorem is true for all forms of $\phi(x)$. M. Liouville, I believe, considers his result as only true under a limitation, and this appears to me to be the correct view. There are at least two distinct cases of the theorem, each of which admits of a convenient transformation in the case of n being fractional.

Let us consider the integral $\int dx\, x^{n-1}\phi(x+a)$, assuming in the first instance that $\phi(x)$ may be developed in ascending powers of x, and that the first exponent exceeds -1.

Put $x = -az$, and let $a = \varepsilon^\theta$, then

$$\begin{aligned}\int dx\, x^{n-1}\phi(a+x) &= (-)^n a^n \int dz\, z^{n-1}\phi\{a(1-z)\}\\ &= (-)^n \varepsilon^{n\theta} \int dz\, z^{n-1}\phi\{\varepsilon^\theta(1-z)\}\\ &= (-)^n \varepsilon^{n\theta} \int dz\, z^{n-1}(1-z)^{\frac{d}{d\theta}}\phi(\varepsilon^\theta).\end{aligned}$$

If we take for the limits of z, 0 and 1, the corresponding limits of x will be 0 and $-a$, and the above gives

$$\int_0^{-a} dx\, x^{n-1}\phi(a+x) = (-)^n \varepsilon^{n\theta}\, \frac{\Gamma(n)\,\Gamma\left(\frac{d}{d\theta}+1\right)}{\Gamma\left(\frac{d}{d\theta}+n+1\right)}\,\phi(\varepsilon^\theta) \ldots (29),$$

a result unquestionably true for all positive values of n. Let n be an integer, then

$$\begin{aligned}&\int_0^{-a} dx\, x^{n-1}\phi(a+x)\\ &\quad= (-)^n \varepsilon^{n\theta}\,\Gamma(n)\left\{\left(\frac{d}{d\theta}+n\right)\left(\frac{d}{d\theta}+n-1\right)\ldots\left(\frac{d}{d\theta}+1\right)\right\}^{-1}\phi(\varepsilon^\theta)\\ &\quad= (-)^n\,\Gamma(n)\,\varepsilon^{n\theta}\varepsilon^{-n\theta}\left\{\frac{d}{d\theta}\ldots\left(\frac{d}{d\theta}-n+1\right)\right\}^{-1}\varepsilon^{n\theta}\phi(\varepsilon^\theta)\\ &\quad= (-)^n\,\Gamma(n)\left\{a^n\left(\frac{d}{da}\right)^n\right\}^{-1} a^n\phi(a)\\ &\quad= (-)^n\,\Gamma(n)\left(\frac{d}{da}\right)^{-n}\phi(a) \ldots (30),\end{aligned}$$

a conclusion easily verified by integration, it being observed that the limits of the integrations relative to a are 0 and a. We might obviously regard this theorem as true for fractional values of n; but, without thus extending our assumptions, we may see from the above, that M. Liouville's formula does

not apply when $\phi(x)$ is only susceptible of development in ascending positive powers of x.

Let us now suppose the development of $\phi(x)$ a descending one, and the exponents negative.

Put $x = az$, $a = \varepsilon^\theta$, as before,

$$\int dx\, x^{n-1} \phi(a+x) = \varepsilon^{n\theta} \int dz\, z^{n-1} (1+z)^{\frac{d}{d\theta}} \phi(\varepsilon^\theta) \ldots . (31),$$

wherein $\dfrac{d}{d\theta}$ is of *negative* interpretation. Now, by a well-known theorem, (Gregory's *Examples*, p. 473),

$$\int_0^\infty dz\, z^{n-1} (1+z)^{-r} = \frac{\Gamma(n)\,\Gamma(r-n)}{\Gamma(r)} \ldots \ldots (32),$$

provided that n be positive and less than r. Hence, comparing with the second member of (31) taken between the same limits,

$$\varepsilon^{n\theta} \int_0^\infty dz\, z^{n-1} (1+z)^{\frac{d}{d\theta}} \phi(\varepsilon^\theta) = \varepsilon^{n\theta} \frac{\Gamma(n)\,\Gamma\left(-\dfrac{d}{d\theta} - n\right)}{\Gamma\left(-\dfrac{d}{d\theta}\right)},$$

whence, observing that the limits of x are also 0 and ∞,

$$\int_0^\infty dx\, x^{n-1} \phi(a+x) = \varepsilon^{n\theta} \frac{\Gamma(n)\,\Gamma\left(-\dfrac{d}{d\theta} - n\right)}{\Gamma\left(-\dfrac{d}{d\theta}\right)} \phi(\varepsilon^\theta) \ldots (33)$$

$$= \varepsilon^{n\theta}\, \Gamma(n) \left\{\left(-\frac{d}{d\theta} - 1\right) \ldots \left(-\frac{d}{d\theta} - n\right)\right\}^{-1} \phi(\varepsilon^\theta)$$

$$= (-)^n\, \Gamma(n)\, \varepsilon^{n\theta} \left\{\left(\frac{d}{d\theta} + n\right) \ldots \left(\frac{d}{d\theta} + 1\right)\right\}^{-1} \phi(\varepsilon^\theta)$$

$$= (-)^n\, \Gamma(n) \left(\frac{d}{da}\right)^{-n} \phi(a) \ldots \ldots \ldots \ldots \ldots (34).$$

Hence, it appears that M. Liouville's formula applies when $\phi(x)$ involves only negative exponents in its development, whereof the first with its sign changed exceeds n. The limits of the integrations relative to a in the second member are $\pm\infty$ and a. § 6.

Finally, let us suppose n frational.

The object of investigation being to determine the function under the sign of integration, when the definite integral is given, let us assume that within the supposed limits, $\int dx\, x^{n-1} \phi(a+x) = \psi(a)$, a known function, and that we wish

to ascertain the form of $\phi(a)$. In the case of the ascending development we have from (29), on changing a into ε^{θ},

$$\psi(\varepsilon^{\theta}) = (-)^n \Gamma(n)\, \varepsilon^{n\theta}\, \frac{\Gamma\left(\frac{d}{d\theta}+1\right)}{\Gamma\left(\frac{d}{d\theta}+n+1\right)}\, \phi(\varepsilon^{\theta}),$$

$$\phi(\varepsilon^{\theta}) = (-)^n\, \frac{1}{\Gamma(n)}\, \frac{\Gamma\left(\frac{d}{d\theta}+n+1\right)}{\Gamma\left(\frac{d}{d\theta}+1\right)}\, \varepsilon^{-n\theta}\, \psi(\varepsilon^{\theta}). \quad \ldots\ldots(35)$$

$$= (-)^n\, \frac{1}{\Gamma(n)}\, \varepsilon^{-n\theta}\, \frac{\Gamma\left(\frac{d}{d\theta}+1\right)}{\Gamma\left(\frac{d}{d\theta}-n+1\right)}\, \psi(\varepsilon^{\theta}). \quad \ldots\ldots\ldots(36).$$

Let i be the integer next below n, being 0 when n is a proper fraction,

$$\varepsilon^{-n\theta}\, \frac{\Gamma\left(\frac{d}{d\theta}+1\right)}{\Gamma\left(\frac{d}{d\theta}-n+1\right)}\, \psi(\varepsilon^{\theta}) = \varepsilon^{-n\theta}\, \frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)\ldots\left(\frac{d}{d\theta}-i\right)\, \frac{\Gamma\left(\frac{d}{d\theta}-i\right)}{\Gamma\left(\frac{d}{d\theta}-n+1\right)}\, \psi(\varepsilon^{\theta})$$

$$= a^{-n}\left\{a^{i+1}\left(\frac{d}{da}\right)^{i+1}\right\}\, \frac{1}{\Gamma(i-n+1)}\, \frac{\Gamma\left(\frac{d}{d\theta}-i\right)\Gamma(i-n+1)}{\Gamma\left(\frac{d}{d\theta}-n+1\right)}\, \psi(\varepsilon^{\theta})$$

$$= \frac{a^{i-n+1}}{\Gamma(i-n+1)}\left(\frac{d}{da}\right)^{i+1} \int_0^1 dv\, v^{\frac{d}{d\theta}-i-1}\, (1-v)^{i-n}\, \psi(\varepsilon^{\theta})$$

$$= \frac{a^{i-n+1}}{\Gamma(i-n+1)}\left(\frac{d}{da}\right)^{i+1} \int_0^1 dv\, v^{-i-1}\, (1-v)^{i-n}\, \psi(av),$$

$$\therefore \phi(a) = (-)^n\, \frac{a^{i-n+1}}{\Gamma(n)\Gamma(i-n+1)}\left(\frac{d}{da}\right)^{i+1} \int_0^1 dv\, v^{-i-1}(1-v)^{i-n}\psi(av) \ldots(37).$$

Similarly for the descending development of $\phi(x)$, and for negative exponents, we have, by (33),

$$\psi(\varepsilon^{\theta}) = \varepsilon^{n\theta}\, \frac{\Gamma(n)\,\Gamma\left(-\frac{d}{d\theta}-n\right)}{\Gamma\left(-\frac{d}{d\theta}\right)}\, \phi(\varepsilon^{\xi})$$

$$\phi(\varepsilon^\theta) = \frac{1}{\Gamma(n)} \frac{\Gamma\left(-\frac{d}{d\theta}\right)}{\Gamma\left(-\frac{d}{d\theta}-n\right)} \varepsilon^{-n\theta}\, \psi(\varepsilon^\theta) \ldots\ldots (38)$$

$$= \frac{1}{\Gamma(n)} \varepsilon^{-n\theta} \frac{\Gamma\left(-\frac{d}{d\theta}+n\right)}{\Gamma\left(-\frac{d}{d\theta}\right)} \psi(\varepsilon^\theta).$$

Now let i be the integer next below n, then

$$\Gamma\left(-\frac{d}{d\theta}+n\right) = \left(-\frac{d}{d\theta}+n-1\right)..\left(-\frac{d}{d\theta}+n-i\right)\Gamma\left(-\frac{d}{d\theta}+n-i\right)$$

$$\Gamma\left(-\frac{d}{d\theta}\right) = \left(-\frac{d}{d\theta}\right)^{-1}\Gamma\left(-\frac{d}{d\theta}+1\right),$$

$$\therefore \phi(\varepsilon^\theta)$$

$$= \frac{1}{\Gamma(n)} \varepsilon^{-n\theta}\left(-\frac{d}{d\theta}+n-1\right)..\left(-\frac{d}{d\theta}+n-i\right)\left(-\frac{d}{d\theta}\right)\frac{\Gamma\left(-\frac{d}{d\theta}+n-i\right)}{\Gamma\left(-\frac{d}{d\theta}+1\right)}$$

$$= (-)^{i+1}\frac{1}{\Gamma(n)}\varepsilon^{-n\theta}\left(\frac{d}{d\theta}-n+i\right)..\left(\frac{d}{d\theta}-n+1\right)\frac{d}{d\theta}\frac{\Gamma\left(-\frac{d}{d\theta}+n-i\right)}{\Gamma\left(-\frac{d}{d\theta}+1\right)}\psi(\varepsilon^\theta)$$

$$= (-)^{i+1}\frac{1}{\Gamma(n)}\varepsilon^{-n\theta}\varepsilon^{(n-i)\theta}\frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)..\left(\frac{d}{d\theta}-i+1\right)\varepsilon^{(i-n)\theta}\frac{d}{d\theta}\frac{\Gamma\left(-\frac{d}{d\theta}+n-i\right)}{\Gamma\left(-\frac{d}{d\theta}+1\right)}\psi(\varepsilon^\theta$$

$$= (-)^{i+1}\frac{1}{\Gamma(n)}\left(\frac{d}{da}\right)^i a^{i-n+1}\frac{d}{da}\frac{1}{\Gamma(i-n+1)}\frac{\Gamma\left(-\frac{d}{d\theta}+n-i\right)\Gamma(i-n+1)}{\Gamma\left(-\frac{d}{d\theta}+1\right)}\psi(\varepsilon^\theta$$

$$= (-)^{i+1}\frac{1}{\Gamma(n)\,\Gamma(i-n+1)}\left(\frac{d}{da}\right)a^{i-n+1}\frac{d}{da}\int_0^1 dv\, v^{n-i-1}(1-v)^{i-n}\psi\left(\frac{a}{v}\right)\ldots.(39)$$

since $v^{-\frac{d}{d\theta}}\psi(\varepsilon^\theta) = \psi\left(\frac{a}{v}\right)$.

We shall somewhat simplify the above formula by changing i into $i-1$, and in (30) and (37), x into $-x$, in order to avoid the negative limit. Our results will then be comprised in the following theorems.

1st. If, n being a positive fraction, $\int_0^a dx\, x^{n-1}\phi(a-x)=\psi(a)$, and if i being the integer next above n, $i-1$ be less than the first exponent in the ascending development of $\psi(a)$, then

$$\phi(a)=\frac{a^{i-n}}{\Gamma(n)\,\Gamma(i-n)}\left(\frac{d}{da}\right)^i\int_0^1 dv\, v^{-i}(1-v)^{i-n-1}\psi(av);$$

but if n be a positive integer less than the same first exponent,

$$\phi(a)=\frac{1}{\Gamma(n)}\left(\frac{d}{da}\right)^n\psi(a).$$

2nd. If, n being a positive fraction, $\int_0^\infty dx\, x^{n-1}\phi(a+x)=\psi(a)$, the exponents in the development of $\psi(a)$ being all negative, and if i be the integer next above n, then

$$\phi(a)=(-)^i\frac{1}{\Gamma(n)\,\Gamma(i-n)}\left(\frac{d}{da}\right)^{i-1}a^{i-n}\,\frac{d}{da}\int_0^1 dv\, v^{n-i}(1-v)^{i-n-1}\,\psi\left(\frac{a}{v}\right);$$

but if n be a positive integer,

$$\phi(a)=(-)^n\frac{1}{\Gamma(n)}\left(\frac{d}{da}\right)^n\psi(a).$$

The application of these formulæ, as it presents no difficulties, so it requires no illustration. They enable us, whenever $\psi(a)$ satisfies the necessary conditions, to determine $\phi(a)$ either in the form of a definite integral, or in that of a series; but for the latter object it is perhaps simpler, changing x into ε^θ, to employ at once the untransformed equations (35) and (38).

I apprehend that the same principles may be extended, by aid of the theorem of M. Dirichlet, to the inverse calculus of definite multiple integrals. At present, however, I have not leisure to pursue this inquiry, nor am I so much concerned to multiply results as to establish principles.

Lincoln, Oct. 26, 1842.

VIII.—ON A NEW SPECIES OF EQUATIONS OF DIFFERENCES.

THE sort of equation to which the heading refers is that in which the degree of the equation with respect to one variable is defined by the other, as in $u_{x,y}=u_{x-y,\,x+y}$, where $u_{x,y}$ is a function of the integers x and y. A problem of some interest leads to such an equation, and will tend to shew the sort of solution which may be given.

It is well known that the number of ways in which the number n may be put together is 2^{n-1}: thus 4 can be con-

structed in 2^3 ways; namely, 4, 31, 13, 22, 211, 121, 112, 1111. But in this theorem orders are counted; thus, 112 is considered as different from 121, and both from 211. The corresponding problem, namely, to find in how many ways the number n can be constructed, different orders not being considered as different ways, is of much greater difficulty, and I am not aware of any thing having been written upon it.

In the first place, if k be not less than n, the number of ways in which n can be put together is the same as that in which $n + k$ can be composed of numbers one of which is k: thus 16 can be $8 + \ldots$. in as many ways as 8 can be put together. If then $u_{x,y}$ represent the number of ways in which x can be made up of y, and numbers not exceeding y, the problem is solved when $u_{x,y}$ is found: for the number of ways in which x can be put together is $u_{2x+k,\ x+k}$, whatever may be the value of k, from 0 inclusive.

Again, it is obvious that the number of ways in which any number, as 16, can be composed of numbers no higher than 6 and not all lower, or $u_{16,6}$ is the sum of the ways of composing 10 out of numbers not exceeding 1, 2...6 Every way in which 10 can be so formed, gives 16 by adding 6. Hence

$$u_{x,y} = u_{x-y,1} + u_{x-y,2} + \ldots + u_{x-y,y},$$

$$u_{x+1,y+1} = u_{x-y,1} + u_{x-y,2} + \ldots + u_{x-y,y+1}.$$

Hence $$u_{x+1,y+1} - u_{x,y} = u_{x-y,y+1},$$

or $$u_{x,y} - u_{x-1,y-1} = u_{x-y,y},$$

an equation of the y^{th} degree with respect to x.

This equation gives a tolerably easy mode of constructing a table: for all numbers up to 10 it is as follows; $u_{x,1}$, $u_{x,x}$, and $u_{x,x-1}$ being always units.

	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	1	1							
4	1	2	1	1						
5	1	2	2	1	1					
6	1	3	3	2	1	1				
7	1	3	4	3	2	1	1			
8	1	4	5	4	3	2	1	1		
9	1	4	7	6	5	3	2	1	1	
10	1	5	8	9	7	5	3	2	1	1

The rule being, after the units are written down, from the y^{th} number of the x^{th} line by adding the $(y-1)^{\text{th}}$ number of the $(x-1)^{\text{th}}$ line to the $(x-y)^{\text{th}}$ number in the x^{th} line. Thus 10 can be formed out of numbers one of which is 4, and no other exceeds 4, in 9 ways: these 9 ways are 442, 4411, 433, 4321, 43111, 4222, 42211, 421111, 4111111.

As soon as we can express $u_{x-1,\,y-1}$ in terms of x and y, we have then an equation of the y^{th} order for determining $u_{x,\,y}$. Say that $u_{x-1,\,y-1} = V_{x,\,y}$, we have then,

$$u_{x,\,y} - u_{x-y,\,y} = V_{x,\,y},$$

which, considering y as a constant, is of the form

$$w_x - w_{x-y} = V_x.$$

From this consideration we can arrive at the general form of the solution, and this is all I shall here attempt. First we have $u_{x,\,1} = 1$, so that

$$u_{x,\,2} - u_{x-2,\,2} = 1;$$

the complete integral of which, the constants being found from $u_{1,\,2} = 0$, $u_{2,\,2} = 1$, is

$$u_{x,\,2} = \frac{x}{2} - \frac{1}{4} + \frac{1}{4}(-1)^x.$$

Therefore $$u_{x,\,3} - u_{x-3,\,3} = \frac{x-1}{2} - \frac{1}{4} + \frac{1}{4}(-1)^{x-1};$$

the complete integral of which, determining the constants from $u_{1,\,3} = 0$, $u_{2,\,3} = 0$, $u_{3,\,3} = 1$, is

$$u_{x,\,3} = \frac{6x^2 - 7 - 9(-1)^x + 8(\beta^x + \gamma^x)}{72},$$

where β and γ are the imaginary cube roots of unity. This may be expressed as follows:—The number of ways in which x can be composed of numbers, none greater and not all less than 3, is

$$\frac{x^2}{12},\ \frac{x^2-1}{12},\ \frac{x^2-4}{12},\ \frac{x^2+3}{12},\ \frac{x^2-4}{12},\ \text{or}\ \frac{x^2-1}{12},$$

according as x divided by 6 leaves a remainder 0, 1, 2, 3, 4, or 5. If we were now to attempt to generalise, we might suppose that

$$u_{x,\,y} \text{ is of the form } A_{y-1} + P_2 + \ldots + P_y,$$

where A_n represents a rational and integral function of the n^{th} degree, and P_n is a periodic or circulating function of the n^{th} order, or going through a cycle of n values. But this

inference would not be quite correct. The terms of each successive integration are of the form

$$\text{definite constant} \times \mu^x \Sigma(\mu^{-x} V_x), \quad (\mu^n = 1).$$

When V_x is a periodic term of the form $C\beta^x$, β being some previous root of unity (say the m^{th}), generally speaking $\mu^x \Sigma . \mu^{-x} V_x$ is a simple circulating function of the m^{th} kind; and if V_x be of the form $A_n \beta^x$, the form resulting from integration is also $A_n \beta^x$. But if $\beta = \mu$, which happens among the cases arising when n is a multiple of m, then $\mu^x \Sigma . \mu^{-x} V_x$ takes the form $\mu^x A_{n+1}$, or the function of x is raised a unit in dimension. Hence the most general form of the solution of $u_{x,y} - u_{x-1,y-1} = u_{x-y,y}$, is

$$u_{x,y} = A_{y-1} + A_{a_2} P_2 + A_{a_3} P_3 + \ldots . A_{a_y} P_y,$$

where a_n, the dimension of the multiplier of P_n, is the integer in $n - y$ divided by y.

The next step gives for $u_{x,4}$ the denominator 864, and the following numerator:

$$6x^3 + 18x^2 - 27x - 39 + (27x + 27)(-1)^x + 32(\beta^{x-1} + \gamma^{x-1} - \beta^x - \gamma^x) + 54\{(\sqrt{-1})^x + (-\sqrt{-1})^x\},$$

this is the 144^{th} part of

(0) $x^3 + 3x^2$,	(6) $x^3 + 3x^2 - 36$,
(1) $x^3 + 3x^2 - 9x + 5$,	(7) $x^3 + 3x^2 - 9x + 5$,
(2) $x^3 + 3x^2 - 20$,	(8) $x^3 + 3x^2 + 96$,
(3) $x^3 + 3x^2 - 9x - 27$,	(9) $x^3 + 3x^2 - 9x - 27$,
(4) $x^3 + 3x^2 + 32$,	(10) $x^3 + 3x^2 - 4$,
(5) $x^3 + 3x^2 - 9x - 11$,	or (11) $x^3 + 3x^2 - 9x - 11$.

according as x divided by 12 gives the remainder 0, 1, 2. . . . or 11.

It thus appears that $u_{x,3}$ may be described as the integer nearest to $x^2 \div 12$, and $u_{x,4}$ as the integer nearest to $(x^3 + 3x^2) \div 144$ or $(x^3 + 3x^2 - 9x) \div 144$, according as x is even or odd. Probably this simple species of description might be continued.

IX.—NOTES ON MAGNETISM. NO. I.

By R. L. Ellis, M.A. Fellow of Trinity College.

A GEOMETRICAL construction, by means of which the action of a small magnet on a distant particle of free magnetism may be readily determined, is mentioned in a memoir by Weber, on the Bifilar Magnetometer (*Scientific Memoirs*, II. p. 270). It is due to Gauss, but I do not know where he has demon-

strated it. A proof of it may be acceptable to some readers of the *Journal.*

I begin by enunciating the construction in question, which will be easily understood without a figure. Let AB be a small bar magnet, c its centre, P the particle of magnetism on which it acts. Join cP, draw PD perpendicular to it, meeting cP produced in D. Let $cQ = \frac{1}{3}cD$. Join PQ. Then PQ or QP (according to the sign of the magnetism of P and the direction of the poles of AB) is the direction of the action of AB on P; and $\frac{Mm}{cP^3}\frac{PQ}{cQ}$ is its magnitude, M being the measure of the magnetism of AB, m that of the magnetism of P.

The dimensions of the magnet being small, and its length in the direction of its axis being much greater than its breadth or thickness, we may proceed as follows.

Conceive the magnet to be composed of a series of intense particles ranged along its axis. Let s be the distance of any one of them from c, μds the measure of its magnetism. Also let $cP = r$, and call the angle cP makes with cD, θ.

Then the distance of the particle in question from P, is $(r^2 + s^2 - 2rs\cos\theta)^{\frac{1}{2}}$, and consequently its action on P is

$$\frac{m\mu ds}{r^2 + s^2 - 2rs\cos\theta},$$

magnetic attraction being supposed to follow the ordinary law. The component of this action along cD, is

$$\frac{m\mu ds}{(r^2 + s^2 - 2rs\cos\theta)^{\frac{3}{2}}}(r\cos\theta - s).$$

This is approximately equal to

$$\frac{m\mu}{r^3}\left(1 + 3\frac{s}{r}\cos\theta\right)(r\cos\theta - s)\,ds,$$

or to $$\frac{m\mu}{r^3}\left\{r\cos\theta + (3\cos^2\theta - 1)\,s\right\}ds;$$

and therefore the total action of AB on P, parallel to cD, is

$$\frac{m\cos\theta}{r^2}\int\mu ds + m\frac{3\cos^2\theta - 1}{r^3}\int\mu s ds,$$

the integrals being taken along the whole length of the magnet. Consequently $\int\mu ds = 0$,

since the aggregate magnetism of a magnet, each element being taken with its proper sign, is zero. Again, $\int\mu s ds$

being the *moment* of the magnetism of AB, is the measure of its magnetic power, or what we have called M.

Consequently the action parallel to cD is

$$\frac{Mm}{r^3}(3\cos^2\theta - 1) \ldots\ldots\ldots\ldots (1).$$

The action of the element at s, perpendicular to cD, is

$$\frac{m\mu ds}{(r^2 + s^2 - 2rs\cos\theta)^{\frac{3}{2}}} r\sin\theta,$$

or, approximately,

$$\frac{m\mu}{r^2}\left(1 + 3\frac{s}{r}\cos\theta\right)\sin\theta ds.$$

The total action perpendicular to cD is, therefore,

$$\frac{3Mm}{r^3}\sin\theta\cos\theta \ldots\ldots\ldots\ldots\ldots (2).$$

The equation to the resultant of these two forces is, since the line passes through P,

$$\frac{y - r\sin\theta}{3\sin\theta\cos\theta} = \frac{x - r\cos\theta}{3\cos^2\theta - 1}.$$

For $y = 0$, or at the point Q,

$$x_0 = r\cos\theta - \tfrac{1}{3} r\sec\theta(3\cos^2\theta - 1),$$

$$\text{or} \quad x_0 = \tfrac{1}{3} r\sec\theta.$$

Now $cD = r\sec\theta$, therefore

$$cQ = \tfrac{1}{3} cD,$$

which proves the first part of the construction.

Next, to find the magnitude of the whole action, square and add (1) and (2), then

$$\frac{Mm}{r^3}(1 - 6\cos^2\theta + 9\cos^4\theta + 9\sin^2\theta\cos^2\theta)^{\frac{1}{2}},$$

$$\text{or} \quad \frac{Mm}{r^3}(1 + 3\cos^2\theta)^{\frac{1}{2}}$$

is the magnitude sought. Now

$$PQ = r\{\sin^2\theta + (\cos\theta - \tfrac{1}{3}\sec\theta)^2\}^{\frac{1}{2}}$$

$$= r(1 - \tfrac{2}{3} + \tfrac{1}{9}\sec^2\theta)^{\frac{1}{2}}$$

$$= \tfrac{1}{3} r(\sec^2\theta + 3)^{\frac{1}{2}},$$

and $$cQ = \tfrac{1}{3} r\sec\theta.$$

Therefore $$\frac{PQ}{cQ} = (1 + 3\cos^2\theta)^{\frac{1}{2}}.$$

Consequently, if R be the magnitude of the resultant sought,

$$R = \frac{Mm}{cP^3} \frac{PQ}{cP},$$

which was to be proved.

The preceding formulæ enable us to determine all the circumstances of the mutual action of two magnets, which are such as to fulfil the conditions of our hypothesis.

For instance, in the memoir *Intensitas vis Magneticæ Terrestris*, Gauss has shown that if a magnet and a needle be placed at right angles to one another, then the moment of rotation of the needle due to the action of the magnet is approximately twice as great when the line of the axis of the magnet passes through the centre of the needle, as it is when the line of the axis of the needle passes through the centre of the magnet.

In neither case has the transverse force (2) any tendency to produce rotation. Its effect is destroyed by the fixed centre of the needle.

Let y be the distance of any element of the needle from the centre, mdy its magnetism. Then, in the first case, we shall have $\theta = \frac{y}{R}$ approximately, (R being the distance between the centre of the magnet and that of the needle). Consequently the force (1) may be expressed by the formula

$$\frac{Mmdy}{(R^2 + y^2)^{\frac{3}{2}}} \left(2 - 3\frac{y^2}{R^2}\right) \text{ since } \cos^2\theta = 1 - \theta^2 \text{ nearly};$$

which, neglecting y^2, becomes

$$\frac{2Mmdy}{R^3};$$

the moment of this round the centre of the needle is $\frac{2Mmydy}{R^3}$, and the total moment is, therefore,

$$\frac{2MM'}{R^3} \quad \text{where } M' = \int mydy.$$

In the second case, $\theta = 0$ for every element of the needle, and the moment sought is, therefore,

$$\frac{MM'}{R^3}, \text{ since (1) then becomes } -\frac{Mm}{r^3},$$

and the sign is immaterial. The moment in this case is, therefore, one half of what it was before, which was to be proved.

This result is included in the following general investigation, in which we shall ascertain the moment of rotation due

to the action of one magnet or needle upon another, whatever be their relative positions.

The data by which we shall suppose the position of the magnets to be determined, are the distance between their centres, the angles which the axis of each respectively makes with the line joining their centres, and the angle between the axes themselves. The last element may be readily replaced by the dihedral angle between two planes which intersect in the line joining the centres of the magnets, and one of which passes through the axis of one of the magnets while the other passes through that of the other.

Let the axis of the magnet, whose action on the other is to be calculated, be taken as axis of x, the centre of the magnet being the origin of co-ordinates. Let x, y, z be the co-ordinates of any point in the other magnet. Then, if $r^2=x^2+y^2+z^2$,

$$3\cos^2\theta - 1 = \frac{3x^2 - r^2}{r^2}, \quad \text{and} \quad \sin\theta\cos\theta = \frac{x\sqrt{(y^2+z^2)}}{r^2}.$$

Again, let a, b, c be the co-ordinates of the centre of the second magnet, α, β, γ the cos angles its axis makes with the co-ordinate axes, ρ the distance of any element $d\rho$ from the centre, X, Y, Z the forces on $d\rho$ parallel to the axes of co-ordinates, m the intensity of $d\rho$. Then

$$X = \frac{Mm}{r^5}(3x^2 - r^2)\,d\rho,$$

$$Y = \frac{3Mm}{r^5}\,xy\,d\rho,$$

$$Z = \frac{3Mm}{r^5}\,xz\,d\rho.$$

Let G, H, K be the moments of these forces about lines drawn through the centre of the second magnet parallel to the axes of co-ordinates,

$$G = \frac{3Mm}{r^5}\,x\,\{y(z-c) - z(y-b)\}\,d\rho,$$

$$H = \frac{Mm}{r^5}\,\{3xz\,(x-a) - (3x^2 - r^2)(z-c)\}\,d\rho,$$

$$K = \frac{Mm}{r^5}\,\{(3x^2 - r^2)(y-b) - 3xy\,(x-a)\}\,d\rho.$$

Now $\qquad x = a + \alpha\rho, \quad y = b + \beta\rho, \quad z = c + \gamma\rho.$

Hence, neglecting the square, &c. of ρ,

$$G = \frac{3Mm}{R^5}\,a\,(b\gamma - c\beta)\,\rho d\rho,$$

$$H = \frac{Mm}{R^5} \{3a\alpha a - (3a^2 - R^2)\gamma\} \rho d\rho,$$

$$K = \frac{Mm}{R^5} \{(3a^2 - R^2)\beta - 3ab\alpha\} \rho d\rho,$$

where $R^2 = a^2 + b^2 + c^2$.

Integrating for ρ, we find, for the total moments,

$$(G) = \frac{3MM'}{R^5} a (b\gamma - c\beta),$$

$$(H) = \frac{3MM'}{R^5} a (c\alpha - a\gamma) + \frac{MM'}{R^3} \gamma,$$

$$(K) = \frac{3MM'}{R^5} a (a\beta - b\alpha) - \frac{MM'}{R^3} \beta,$$

M' being for the second magnet what M is for the first.

Let L be the resultant of these three moments, then

$$L^2 = (G)^2 + (H)^2 + (K)^2.$$

Consequently, since $\alpha^2 + \beta^2 + \gamma^2 = 1$, we shall have

$$L^2 = \left(\frac{3MM'}{R^5}\right)^2 a^2 \{R^2 - (a\alpha + b\beta + c\gamma)^2\} + 6 \left(\frac{MM'}{R^4}\right)^2 \{a\alpha (a\alpha + b\beta + c\gamma) - a^2\} + \left(\frac{MM'}{R^3}\right)^2 (\beta^2 + \gamma^2).$$

Now, let θ, θ' be the angles which the axes of the two magnets make respectively with the line joining their centres, and let ϕ be the angle between the axes themselves. Then

$a = R \cos \theta$ and $a\alpha + b\beta + c\gamma = R \cos \theta'$, also $\alpha = \cos \phi$.

Consequently

$$L^2 = \left(\frac{MM'}{R^3}\right)^2 \{9 \cos^2\theta \sin^2\theta' + 6 \cos\theta (\cos\theta' \cos\phi - \cos\theta) + \sin^2\phi\},$$

or $$L = \frac{MM'}{R^3} \{1 + 3 \cos^2 \theta - (3 \cos \theta \cos \theta' - \cos \phi)^2\}^{\frac{1}{2}}.$$

Let χ be the dihedral angle already mentioned, then

$$\cos \phi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \chi.$$

Consequently the last equation becomes

$$L = \frac{MM'}{R^3} \{1 + 3 \cos^2 \theta - (2 \cos \theta \cos \theta' - \sin \theta \sin \theta' \cos \chi)^2\}^{\frac{1}{2}},$$

the required expression.

ADDENDUM TO ART. II.

The method of this paper may be applied to the integration of any linear equation with constant coefficients, by means of the following theorems, the first of which answers to integration by parts, the second to John Bernoulli's Theorem. Let D denote the operation of finding a differential coefficient, then

$$(D+a)^{-1}.PQ = P(D+a)^{-1}Q - (D+a)^{-1}\{P'(D+a)^{-1}Q\},$$

P' being the result of DP: and, abbreviating $D+a$ into Θ,

$$\Theta^{-1}.PQ = P\Theta^{-1}Q - P'\Theta^{-2}Q + P\Theta^{-3}Q\ldots.$$

which, if P be rational and integral, gives $\Theta^{-1}.PQ$ in terms of $\Theta^{-1}Q$, &c.

Thus, to find all the preceding equations of

$$(D-1)^6(D-2)^8y = X,$$

those of the 13th degree, for instance: operate upon

$$(D-1)^{-1}\{x^m(D-1)^6(D-2)^8y\} = (D-1)^{-1}(x^mX),$$

$$(D-2)^{-1}\{x^n(D-2)^8(D-1)^6y\} = (D-2)^{-1}(x^nX),$$

for all integer values of m and n, from $m=0$ to $m=5$, and from $n=0$ to $n=7$. We shall thus have the 14 equations of the 13th degree; and from each of these, exactly as in the preceding part of this paper, others of the 12th, &c. degrees may be obtained.

The criterion which, being satisfied, shows that

$$\Sigma\{P_m(D+a)^my\}$$

will, when operated upon by the symbol $(D+a)^{-1}$, not show that symbol in the result, precisely resembles the corresponding criterion when $a=0$, or the case of simple integration.

The preceding method may be equally applied to finding the intermediate equations of linear partial differential equations with constant coefficients, or of equations of differences.

With respect to equations of differences, the process answering to integration by parts is as follows. Let E signify the operation of changing x into $x+\Delta x$, then, P and Q being functions of x,

$$(E+a)^{-1}.(PQ) = P(E+a)^{-1}Q - (E+a)^{-1}\{\Delta P.(E+a)^{-1}EQ\},$$

and the extension of John Bernoulli's theorem is, $E+a$ being Θ,

$$\Theta^{-1}(PQ) = P\Theta^{-1}Q - \Delta P.\Theta^{-2}EQ + \Delta^2P\Theta^{-3}E^2Q - \ldots.$$

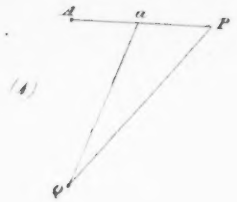
which terminates when P is rational and integral.

In applying this to such an equation as

$$(E-a)^m(E-b)^n\ldots.y + X,$$

instead of using x^k as a multiplier, it would be more convenient to use the factorial $x^{k|}$.

A. DE M.



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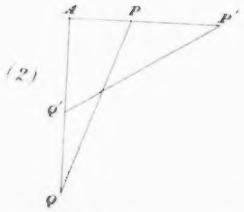

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